Extensive Games with Imperfect Information

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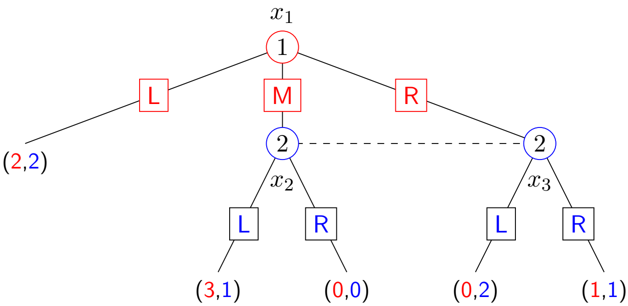
- Here, we look at dynamic games of incomplete information, where the analysis is based on the notions of
  - Weak Perfect Bayesian Nash equilibrium, and
  - Sequential equilibrium.

Example 1



	L	R
L	2 2	2 2
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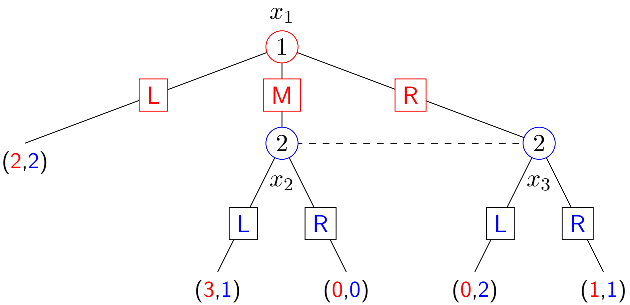

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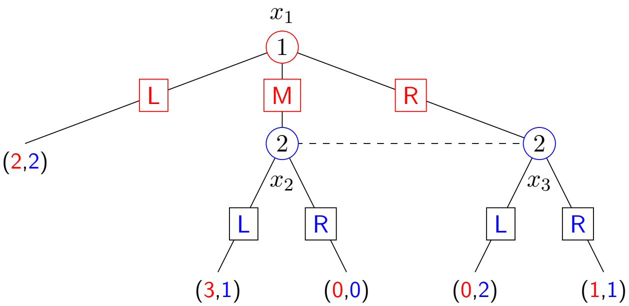

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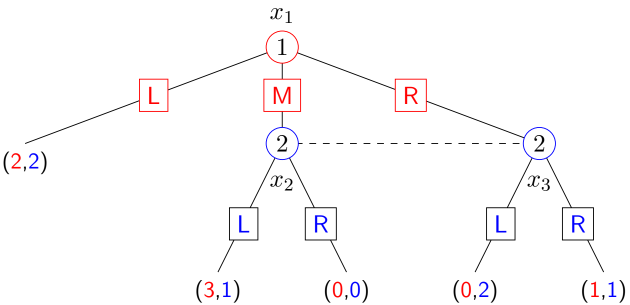

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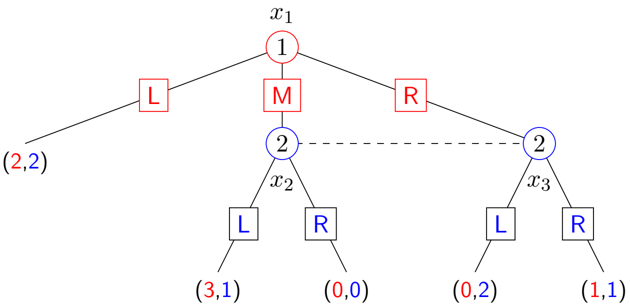

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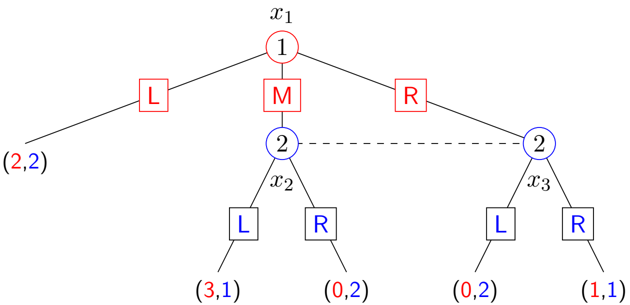

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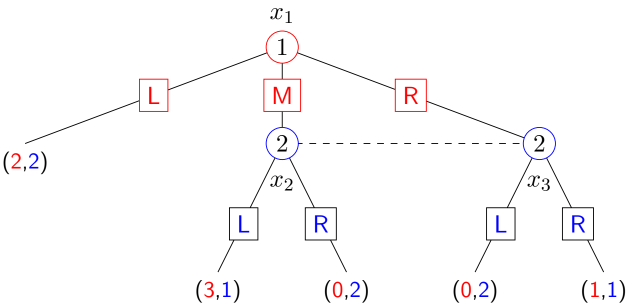

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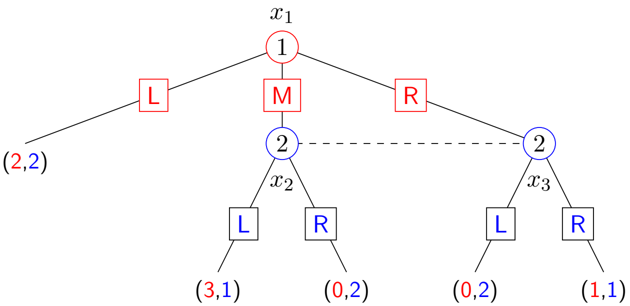

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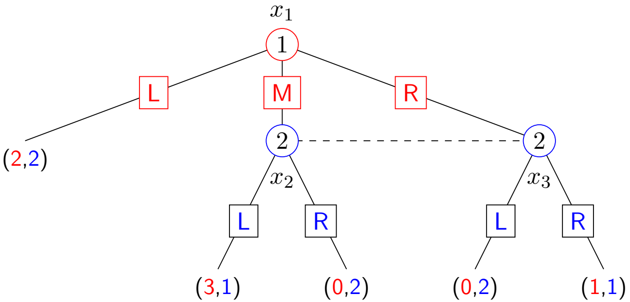

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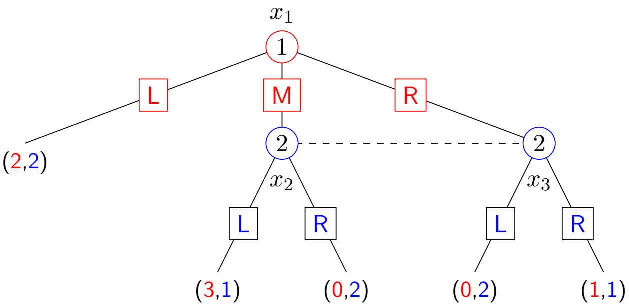

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Extensive Game with Imperfect Information and Chance Moves

An **extensive game with imperfect information and chance moves** consists of:

- a set of **players**,
- a set of sequences (**terminal histories**) with the property that no sequence is a proper subhistory of any other sequence,
- a function (the **player function**) that assigns a player (or chance) to every sequence that is a proper subhistory of some terminal history,
- a function that assigns to each history that the player function assigns to chance a probability distribution over the actions available after that history, where each such distribution is independent of every other such distribution,
- for each player, a partition (the player's **information partition**) of the set of histories assigned to that player by the player function such that for every history h in any given member of the partition, the set $A(h)$ of actions available is the same, and
- for each player, **preferences** over the set of lotteries over terminal histories.

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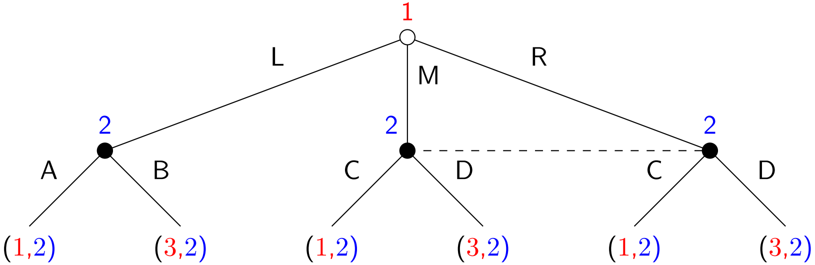
  - Denote $h \in H$ where $h \subset X$.
  - Each node is present in only one information set h, i.e., $x \in h(x)$.
  - When a player is choosing an action at $h(x)$, he cannot tell if it is in $x \in h(x)$ or $x' \in h(x)$.

Information Sets (Example)

Here is an example of a game with non-singleton information sets.



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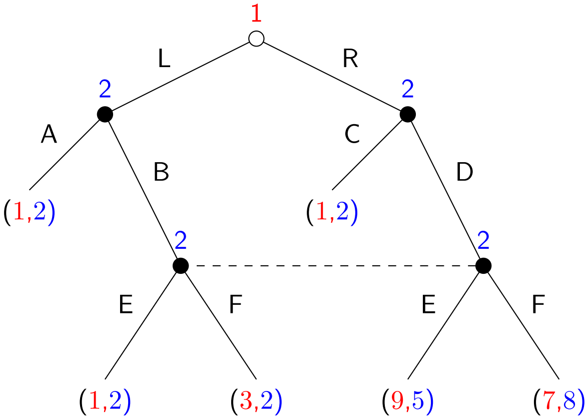
Imperfect Recall (Example)

Here is an example of an extensive-form game with imperfect recall.



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- How many pure strategies are there in the previous examples?

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- The behavioral strategy probabilities are independent at each node.

Equivalence of Mixed and Behavioral Strategies

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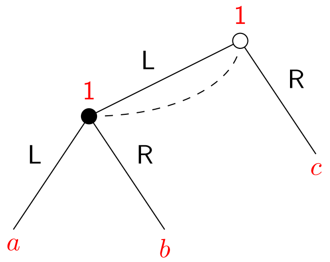

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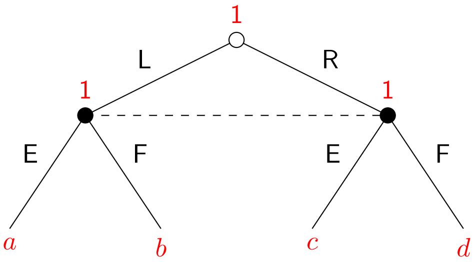

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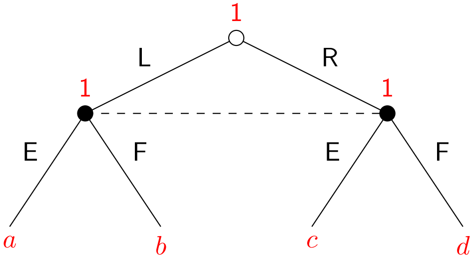

Theorem (Kuhn (1953))

In games with perfect recall, mixed and behavioral strategies are equivalent.

Strategies

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- The expected utility given assessment (σ, μ) and given that the game has reached information set h is

$$u_i \left(\sigma_i, \sigma_{-i}, \mu | h\right).$$

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for all $\sigma'_{i(h)} \in \Sigma_{i(h)}$. If the assessment is sequentially rational for all information sets h of the game G, then, we say that σ is sequentially rational given belief system μ.

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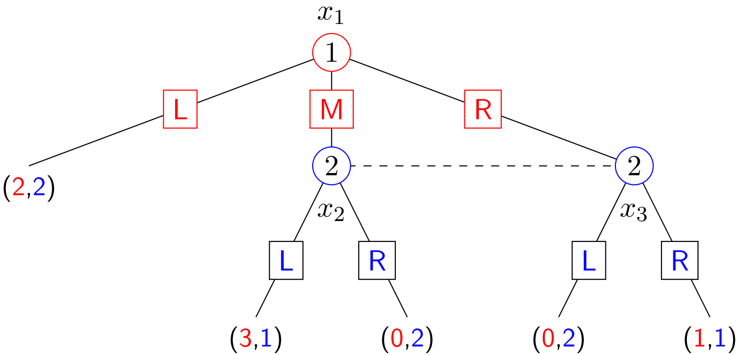
- Given belief system every player's strategy is a best response to all other players' strategies.

Example



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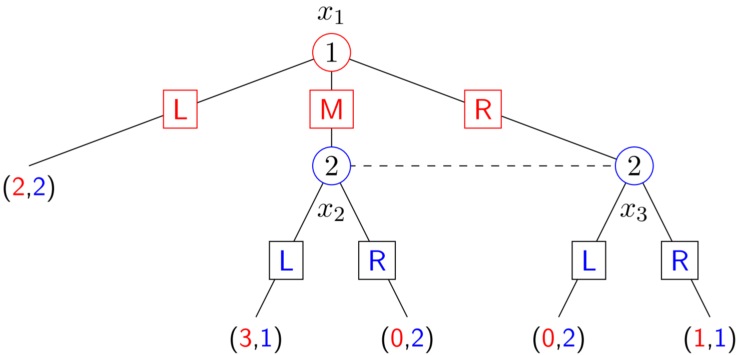

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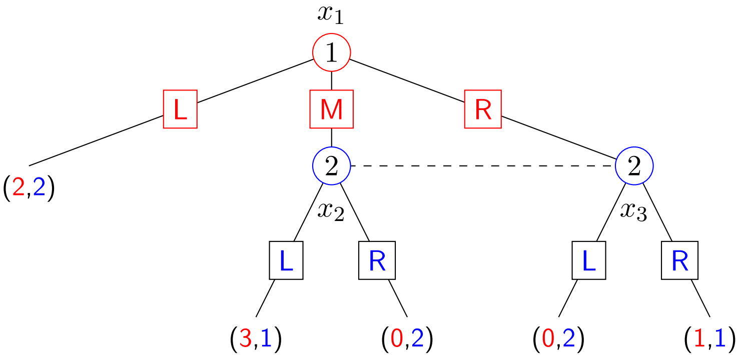

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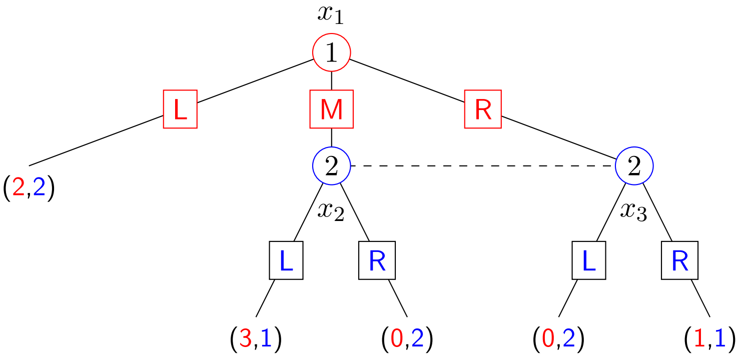

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- $(\sigma_1(M) = 1, \sigma_2(L) = 1, \mu(x_2) < .5)$

Weak Consistency

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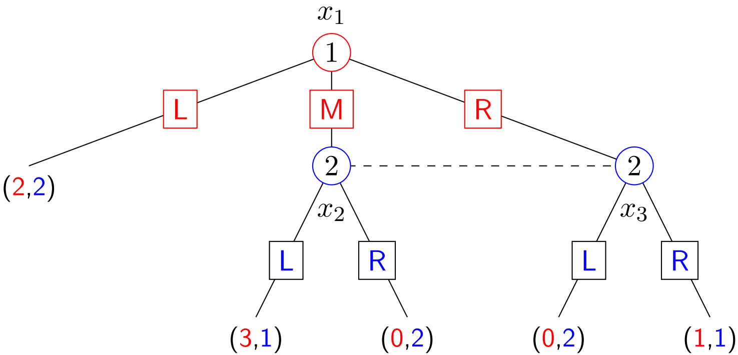
Definition

A belief system μ of assessment (σ, μ) is said to be **weakly consistent** if for any information set h such that $\sum_{x \in h} P(x, \sigma) > 0$,

$$\mu(x) = P(x|h, \sigma) = \frac{P(x|\sigma)}{\sum_{x' \in h} P(x'|\sigma)}.$$

If $\sum_{x \in h} P(x, \sigma) = 0$, then, the only requirement is that $\sum_{x \in h} \mu(x) = 1$.

Example



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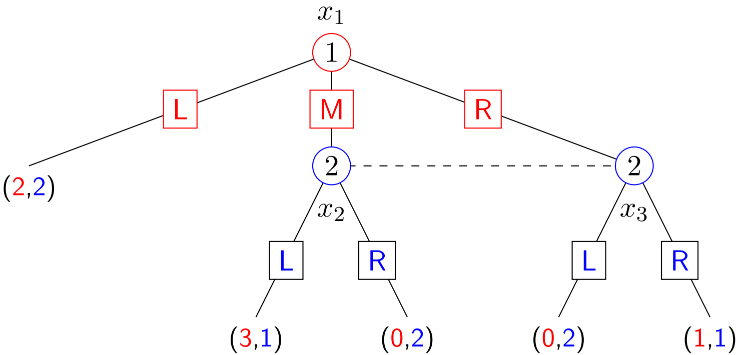

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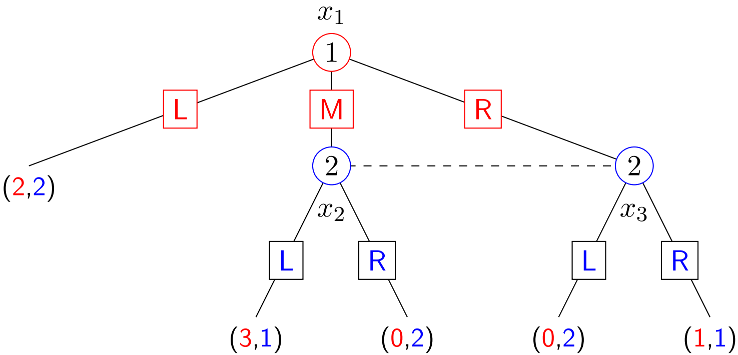

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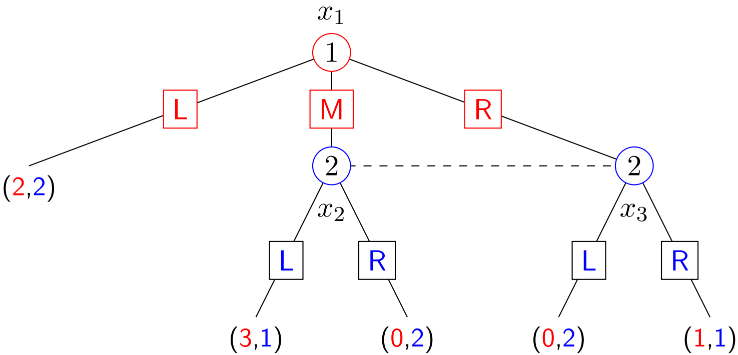

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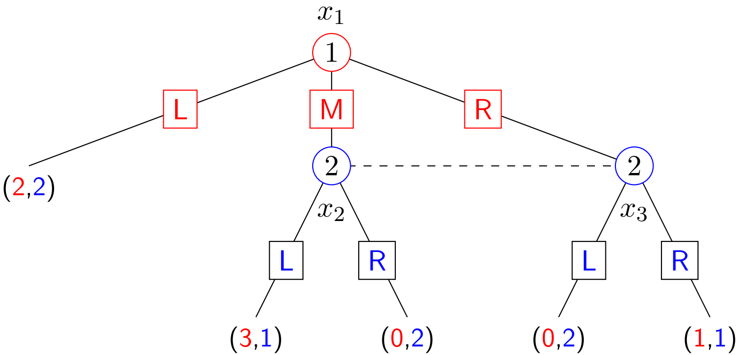

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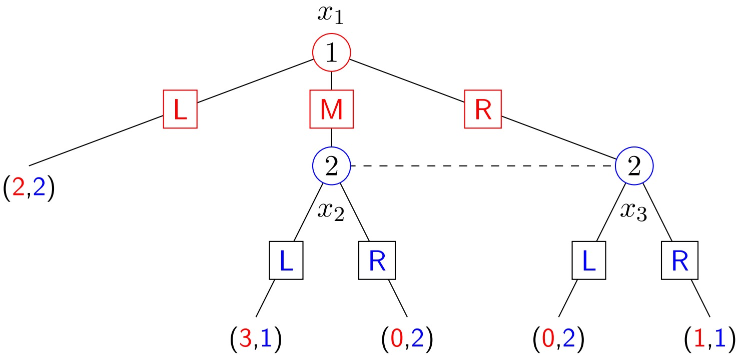

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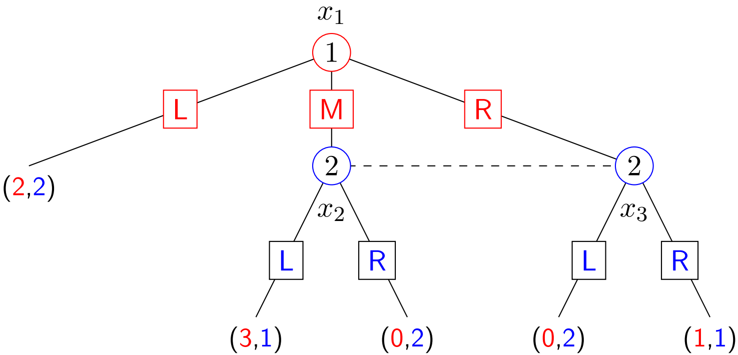

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- If $\sigma_1 = (1, 0, 0)$ and σ_2, then, $\mu(x_2) \in [0, 1]$

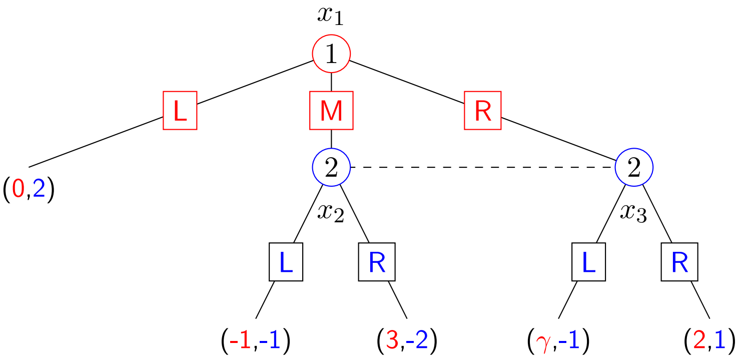
Weak Perfect Bayesian Equilibrium

Definition

An assessment (σ, μ) is a **Weak Perfect Bayesian equilibrium** if

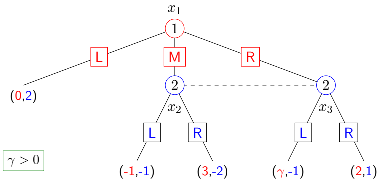
1. the strategy profile σ is sequentially rational given belief system μ, and
2. the belief system μ is weakly consistent given strategy profile σ.

WPBE (Exercise 1)

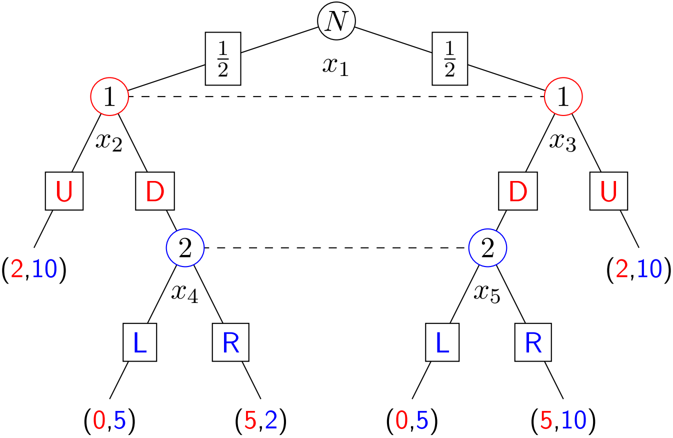


- Assume $\gamma > 0$.

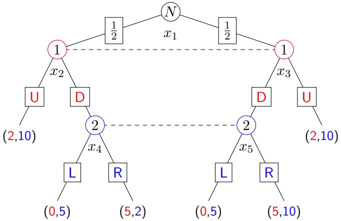
WPBE (Exercise 1) (Cont.)



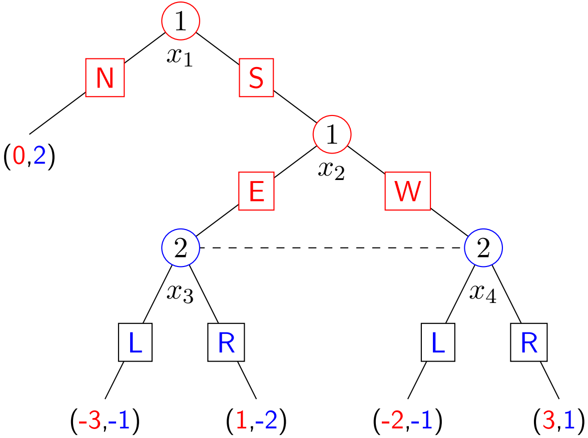
WPBE (Exercise 2)



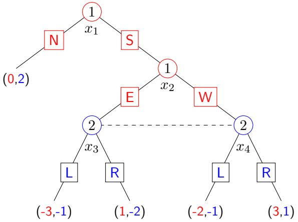
WPBE (Exercise 2) (Cont.)



WPBE (Exercise 3)



WPBE (Exercise 3) (Cont.)



Consistency

- Kreps and Wilson (1982) introduced the notion of consistency.

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Definition

A belief system μ of assessment (σ, μ) is said to be **consistent** if there exists a sequence of completely mixed strategies $\{\sigma^k\}_{k=1}^{\infty}$ with $\lim_{k\to\infty} \sigma^k = \sigma$, such that $\mu = \lim_{k\to\infty} \mu^k$, where μ^k denotes the beliefs derived from strategy profile σ^k using Bayes' rule.

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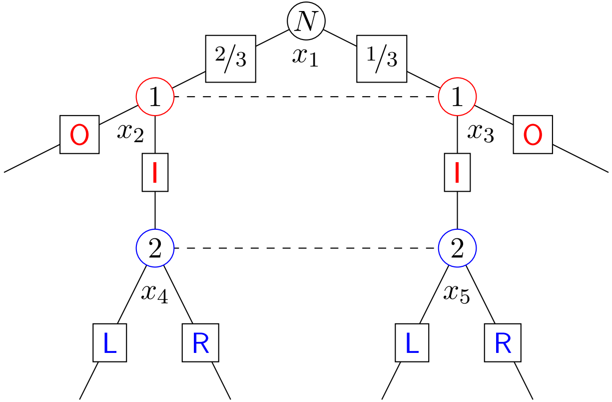
- As opposed to weak consistency, consistency now has requirements for beliefs of probability zero events given σ.

Example



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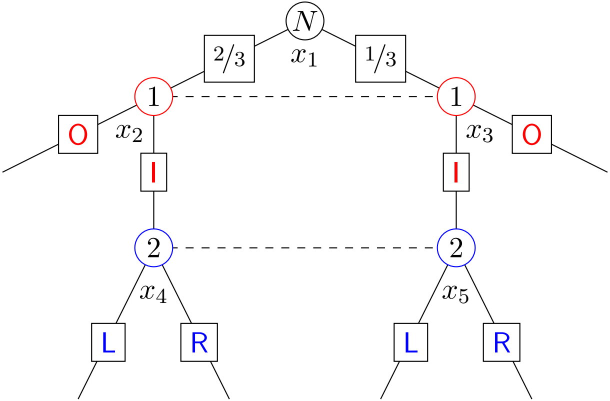
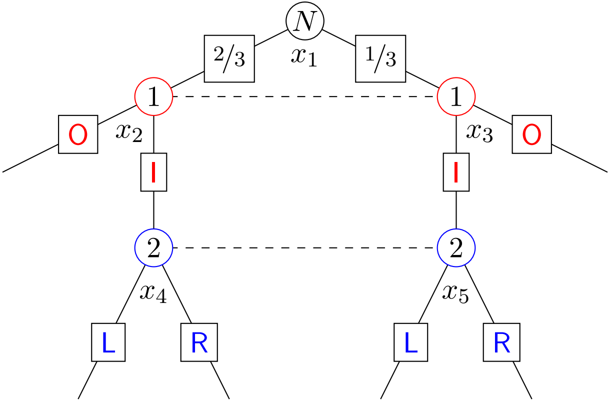


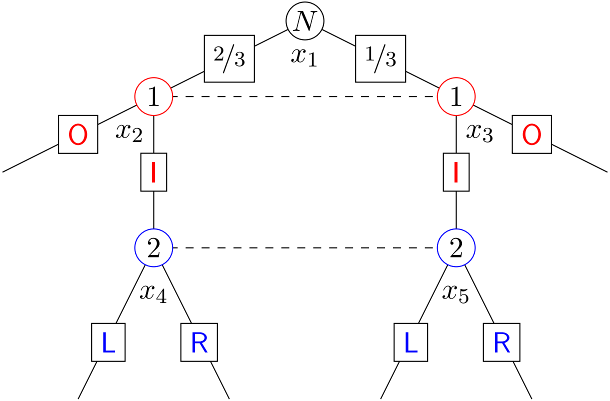
- $\mu\left(x_2\right) = {}^{2}\!/_{3}$

Example



- $\mu(x_2) = 2/3$
- $\sigma(O) = 1$
- Is player 1 as likely to deviate on both sides?

Example



- $\mu(x_2) = {}^2\!/_3$
- $\sigma(O) = 1$
- Is player 1 as likely to deviate on both sides?
- $\mu(x_4) = {}^2\!/_3$

Sequential Equilibrium

Definition

An assessment (σ, μ) is a **Sequential equilibrium** if

1. the strategy profile σ is sequentially rational given belief system μ, and
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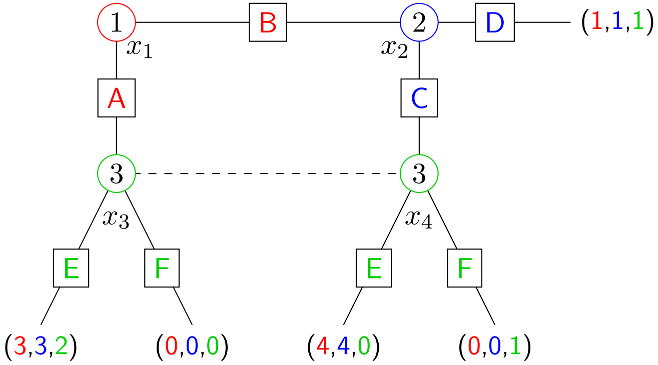
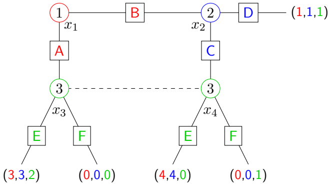
- Every SE is both a WPBE and a SPNE.

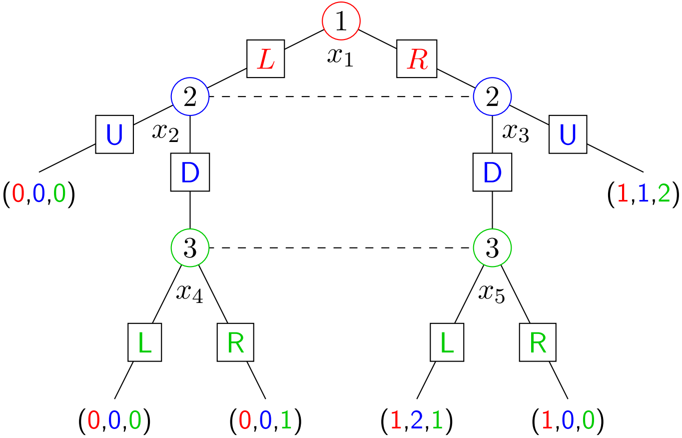
SE (Exercise 1)



SE (Exercise 1) (Cont.)



SE (Exercise 2)



SE (Exercise 2) (Cont.)