Extensive Games with Perfect Information: Extensions and Discussion

Motivation (Example)

- Consider extensive form games with sequential and simultaneous moves.

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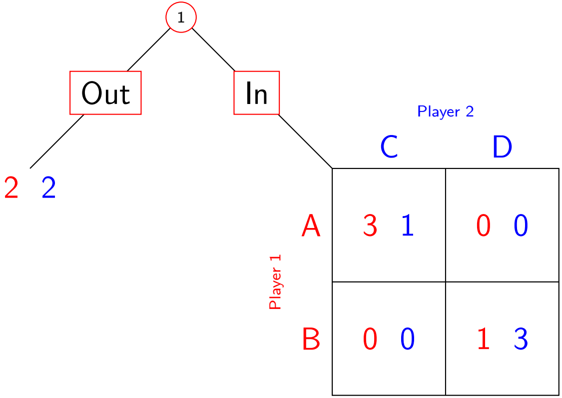

Extensive Game with Perfect Information and Simultaneous Moves

Definition

An **extensive game with perfect information and simultaneous moves** consists of:

- a set of **players**,
- a set of sequences (**terminal histories**) with the property that no sequence is a proper subhistory of any other sequence,
- a function (the **player function**) that assigns a set of players to every sequence that is a proper subhistory of some terminal history,
- for each proper subhistory h of each terminal history and each player i that is a member of the set of players assigned to h by the player function, a set $A_i(h)$ (the set of **actions** available to player i after the history h), and
- **preferences** over the set of terminal histories for each player.

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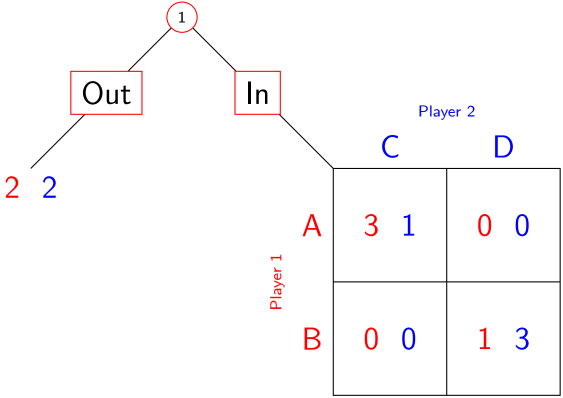
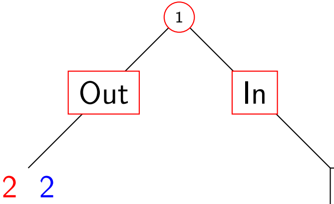
Motivation (Example) (Cont.)



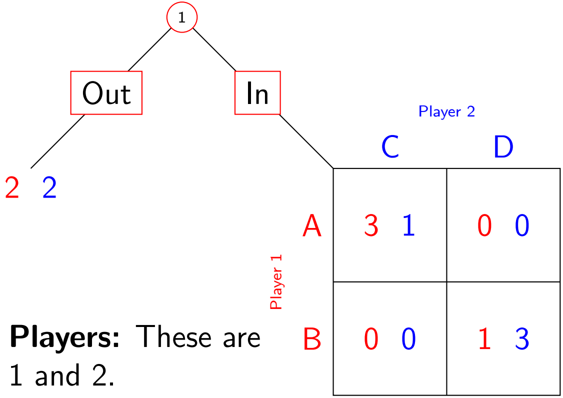
Motivation (Example) (Cont.)



		Player 2: C	Player 2: D
Player 1	A	3 1	0 0
	B	0 0	1 3

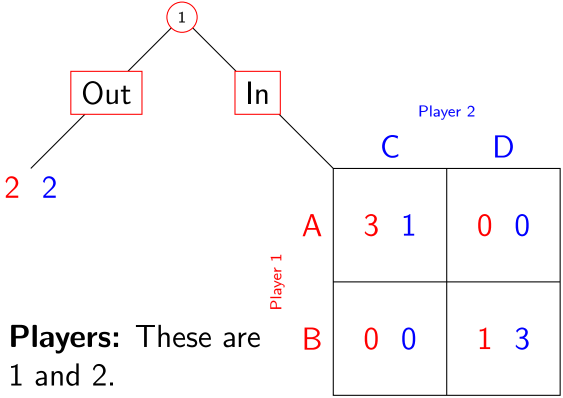
- **Players:**

Motivation (Example) (Cont.)



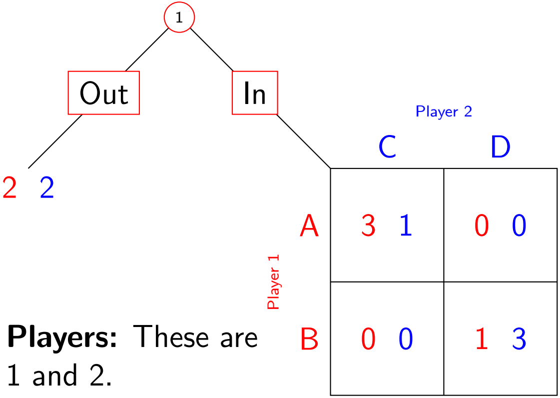
- **Players:** These are 1 and 2.

Motivation (Example) (Cont.)



- **Players:** These are 1 and 2.
- **Terminal Histories:**

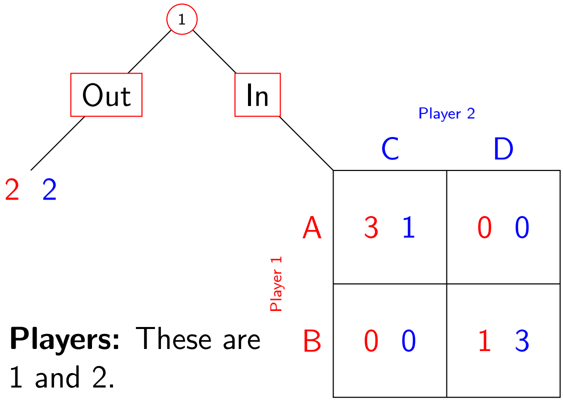
Motivation (Example) (Cont.)



- **Players:** These are 1 and 2.
- **Terminal Histories:**

$$(Out),$$

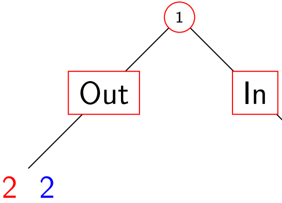
Motivation (Example) (Cont.)

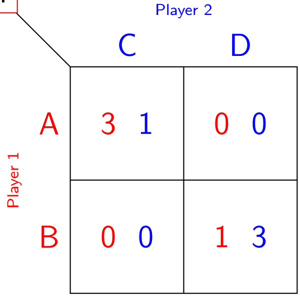


- **Players:** These are 1 and 2.
- **Terminal Histories:**

$$(Out)\,, \quad (In,(A,C))\,,$$

Motivation (Example) (Cont.)

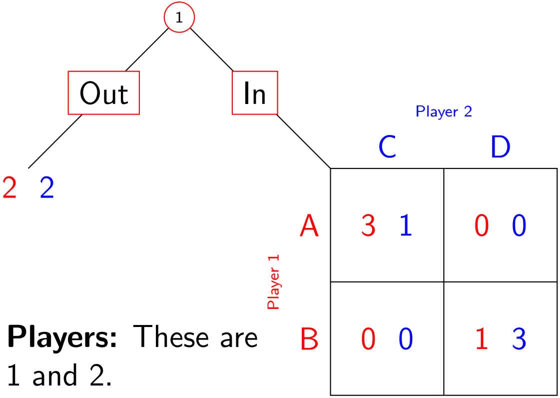




- **Players:** These are 1 and 2.
- **Terminal Histories:**

$$(Out)\,, \quad (In, (A, C))\,, \quad (In, (B, C))\,,$$

Motivation (Example) (Cont.)

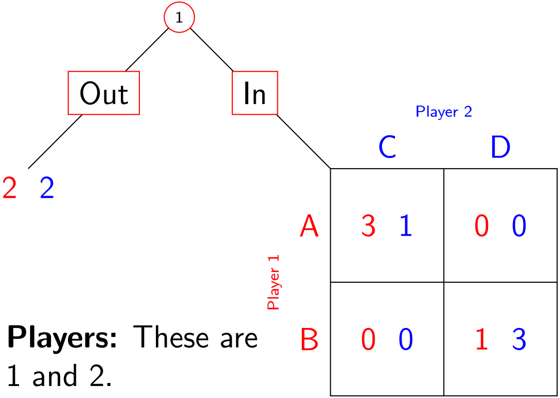


- **Players:** These are 1 and 2.
- **Terminal Histories:**

$$(Out), \quad (In, (A, C)), \quad (In, (B, C)),$$

$$(In, (A, D)),$$

Motivation (Example) (Cont.)

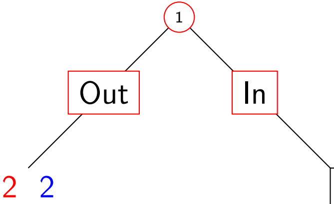


- **Players:** These are 1 and 2.
- **Terminal Histories:**

$$(Out), \quad (In, (A, C)), \quad (In, (B, C)),$$

$$(In, (A, D)), \quad (In, (B, D)).$$

Motivation (Example) (Cont.)



Player 2

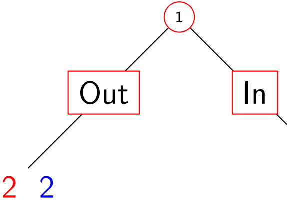
Player 1	C	D
A	3 1	0 0
B	0 0	1 3

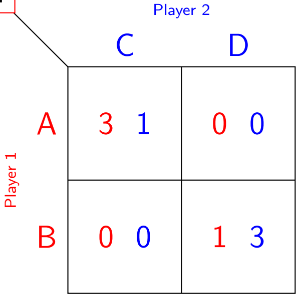
- **Players:** These are 1 and 2.
- **Terminal Histories:**

$$(Out)\,, \quad (In, (A, C))\,, \quad (In, (B, C))\,,$$

$$(In, (A, D))\,, \quad (In, (B, D))\,.$$

Motivation (Example) (Cont.)





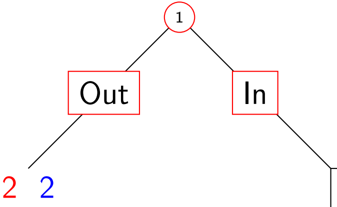
- **Player function:**

- **Players:** These are 1 and 2.
- **Terminal Histories:**

$$(Out)\,,\quad (In,(A,C))\,,\quad (In,(B,C))\,,$$

$$(In,(A,D))\,,\quad (In,(B,D))\,.$$

Motivation (Example) (Cont.)



Player 2

Player 1	C	D
A	3 1	0 0
B	0 0	1 3

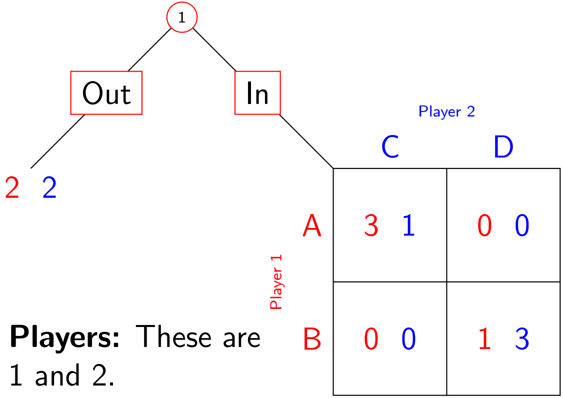
- **Players:** These are 1 and 2.
- **Terminal Histories:**

$$(Out), \quad (In,(A,C)), \quad (In,(B,C)),$$

$$(In,(A,D)), \quad (In,(B,D)).$$

- **Player function:**
  - $P(\emptyset) = \{1\}$,

Motivation (Example) (Cont.)



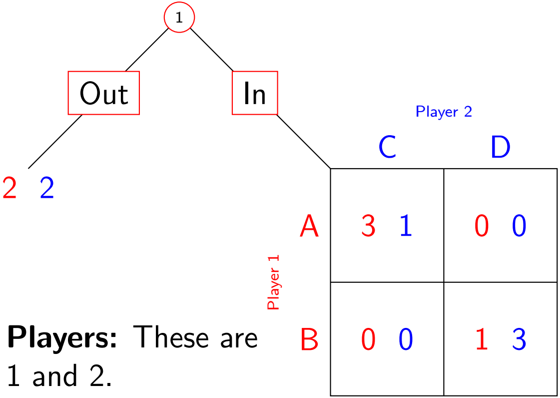
- **Player function:**
  - $P(\emptyset) = \{1\}$,
  - $P(In) = \{1, 2\}$.

- **Players:** These are 1 and 2.
- **Terminal Histories:**

$$(Out), \quad (In, (A, C)), \quad (In, (B, C)),$$

$$(In, (A, D)), \quad (In, (B, D)).$$

Motivation (Example) (Cont.)

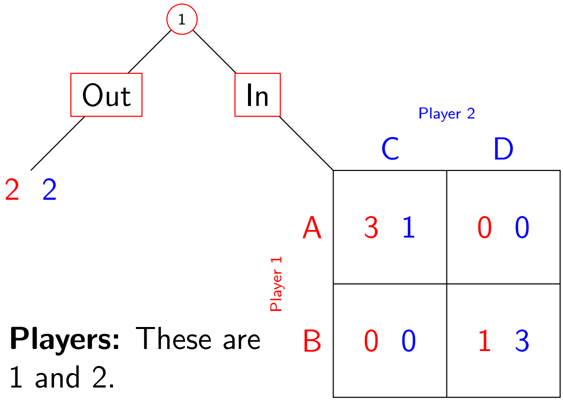


- **Player function:**
  - $P(\emptyset) = \{1\}$,
  - $P(In) = \{1, 2\}$.
- **Actions:**
- **Players:** These are 1 and 2.
- **Terminal Histories:**

$$(Out), \quad (In, (A, C)), \quad (In, (B, C)),$$

$$(In, (A, D)), \quad (In, (B, D)).$$

Motivation (Example) (Cont.)

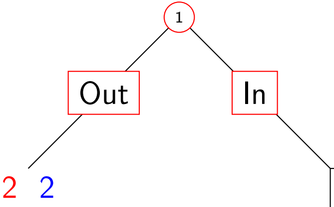


- **Player function:**
  - $P(\emptyset) = \{1\}$,
  - $P(In) = \{1, 2\}$.
- **Actions:**
  - $A_1(\emptyset) = \{In, Out\}$,
- **Players:** These are 1 and 2.
- **Terminal Histories:**

$$(Out), \quad (In, (A, C)), \quad (In, (B, C)),$$

$$(In, (A, D)), \quad (In, (B, D)).$$

Motivation (Example) (Cont.)



Player 2

	C	D
A	3 1	0 0
B	0 0	1 3

Player 1

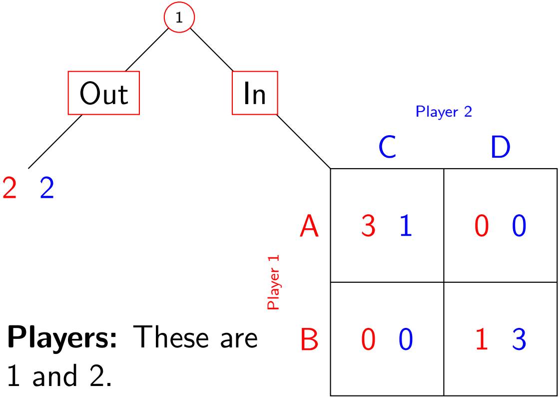
- **Players:** These are 1 and 2.
- **Terminal Histories:**

$$(Out), \quad (In, (A, C)), \quad (In, (B, C)),$$

$$(In, (A, D)), \quad (In, (B, D)).$$

- **Player function:**
  - $P(\emptyset) = \{1\}$,
  - $P(In) = \{1, 2\}$.
- **Actions:**
  - $A_1(\emptyset) = \{In, Out\}$,
  - $A_1(In) = \{A, B\}$,

Motivation (Example) (Cont.)



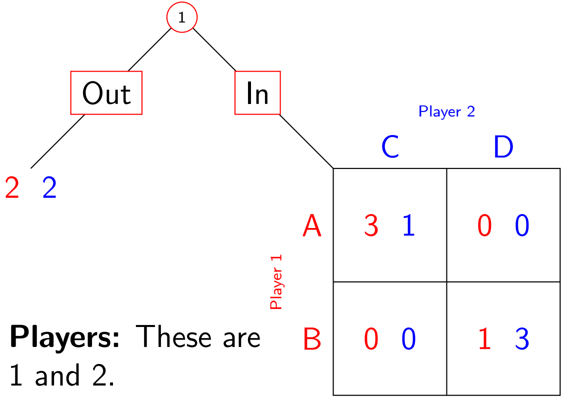
- **Players:** These are 1 and 2.
- **Terminal Histories:**

$$(Out), \quad (In,(A,C)), \quad (In,(B,C)),$$

$$(In,(A,D)), \quad (In,(B,D)).$$

- **Player function:**
  - $P(\emptyset) = \{1\}$,
  - $P(In) = \{1,2\}$.
- **Actions:**
  - $A_1(\emptyset) = \{In, Out\}$,
  - $A_1(In) = \{A, B\}$,
  - $A_2(In) = \{C, D\}$.

Motivation (Example) (Cont.)



- **Player function:**
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- **Actions:**
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- **Preferences:**
- **Players:** These are 1 and 2.
- **Terminal Histories:**

$$(Out), \quad (In, (A, C)), \quad (In, (B, C)),$$

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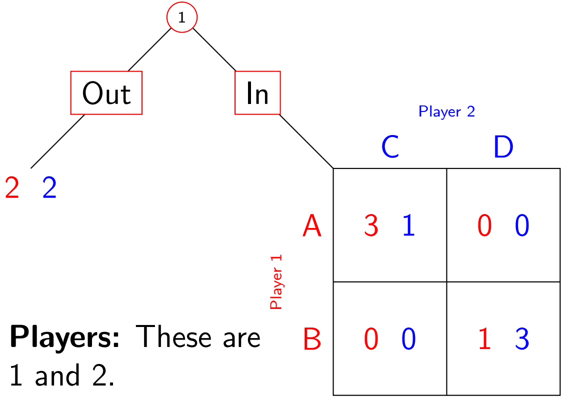
Motivation (Example) (Cont.)



- **Players:** These are 1 and 2.
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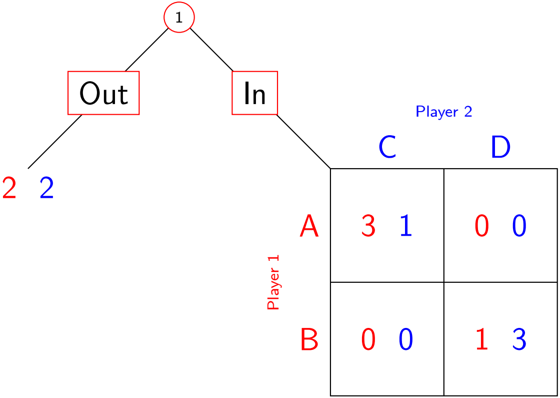
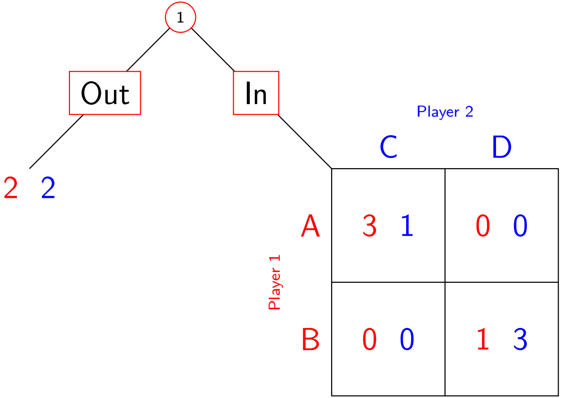
- **Actions:**
  - $A_1(\emptyset) = \{In, Out\}$,
  - $A_1(In) = \{A, B\}$,
  - $A_2(In) = \{C, D\}$.
- **Preferences:**
  - On the tree.

Nash Equilibrium

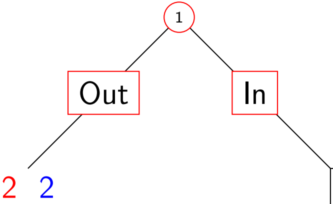


Nash Equilibrium



Out,A		
Out,B		
In,A		
In,B		

Nash Equilibrium



Player 2

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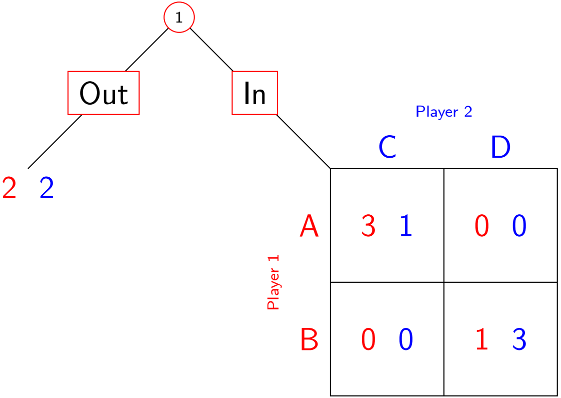
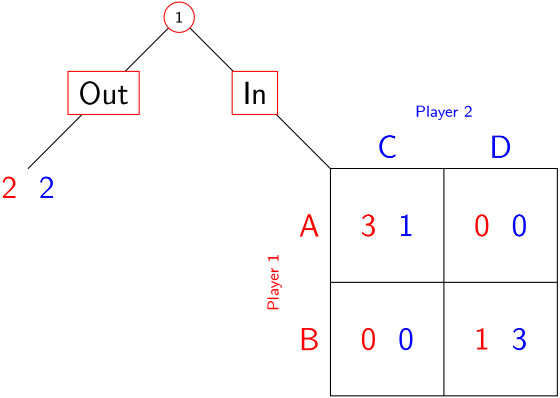
	C	D
Out,A		
Out,B		
In,A		
In,B		

Nash Equilibrium



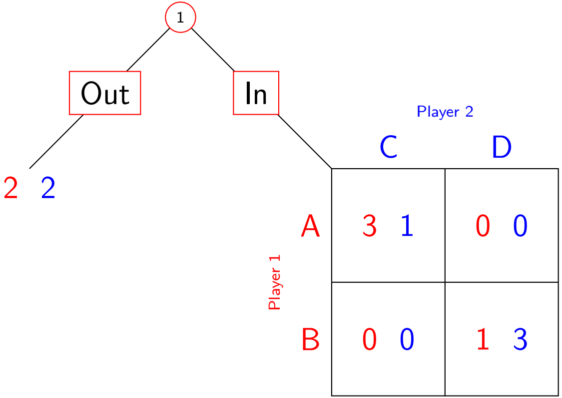
	C	D
Out,A	2 2	2 2
Out,B	2 2	2 2
In,A		
In,B		

Nash Equilibrium

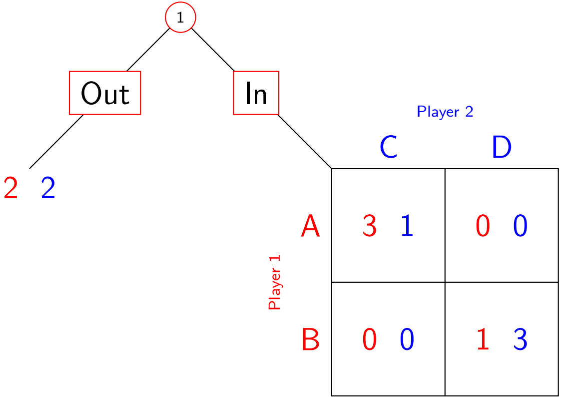


	C	D
Out,A	2 2	2 2
Out,B	2 2	2 2
In,A	3 1	0 0
In,B	0 0	1 3

Subgame Perfect Nash Equilibrium

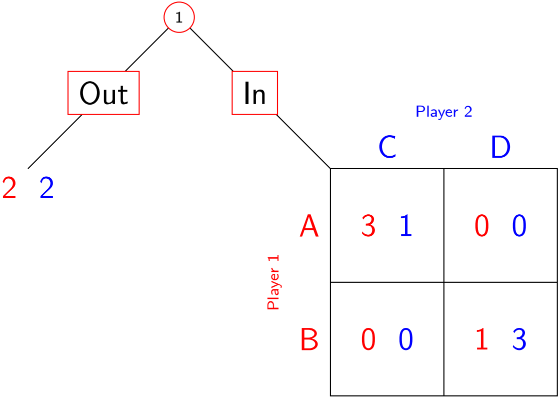


Subgame Perfect Nash Equilibrium



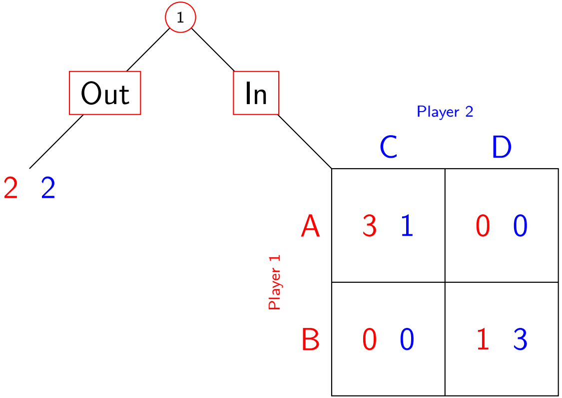
- Recall that a subgame perfect Nash equilibrium must be a Nash equilibrium in every subgame.

Subgame Perfect Nash Equilibrium



- Recall that a subgame perfect Nash equilibrium must be a Nash equilibrium in every subgame.
- How many subgames are there?

Subgame Perfect Nash Equilibrium



- Recall that a subgame perfect Nash equilibrium must be a Nash equilibrium in every subgame.
- How many subgames are there?
- What are the SPNE?

Illustration 1: Entry into Monopolized Industry

- **Players:**
  - Firm 1 is the monopolist, and
  - Firm 2 is the potential entrant.

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- **Players:**
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  - Firm 2 is the potential entrant.
- **Terminal Histories:**
  - (Out, q_1) for $q_1 \in [0, \infty)$,
  - (In,(q_1, q_2)) for $q_i \in [0, \infty)$.

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- **Player Function:**
  - $P(\emptyset) = \{2\}$,
  - $P(Out) = \{1\}$,
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- **Player Function:**
  - $P(\emptyset) = \{2\}$,
  - $P(Out) = \{1\}$,
  - $P(In) = \{1, 2\}$.
- **Actions:**

$$A_2(\emptyset) = \{In, Out\}, \qquad A_1(Out) = q_1 \in [0, \infty),$$

$$A_1(In) = q_1 \in [0, \infty), \qquad A_2(In) = q_2 \in [0, \infty).$$

Illustration 1: Entry into Monopolized Industry (Cont.)

- **Preferences:**
  - The inverse demand function is given by

$$P(Q) = \begin{cases} \alpha - Q & Q < \alpha \\ 0 & Q \geq \alpha. \end{cases}$$

and

Illustration 1: Entry into Monopolized Industry (Cont.)

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- **Preferences:**
  - The inverse demand function is given by

$$P(Q) = \begin{cases} \alpha - Q & Q < \alpha \\ 0 & Q \geq \alpha. \end{cases}$$

  - The cost is given by
    - $C_1(q_1) = cq_1$,
    - $C_2(q_2) = cq_2 + f$.

and

Illustration 1: Entry into Monopolized Industry (cont.)

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  - The inverse demand function is given by

$$P(Q) = \begin{cases} \alpha - Q & Q < \alpha \\ 0 & Q \geq \alpha. \end{cases}$$

  - The cost is given by
    - $C_1(q_1) = cq_1$,
    - $C_2(q_2) = cq_2 + f$.
  - The fixed cost to enter for firm 2 is f.

and

Illustration 1: Entry into Monopolized Industry (Cont.)

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  - The inverse demand function is given by

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  - The cost is given by
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  - The fixed cost to enter for firm 2 is f.
  - The profits of the firms are

and

Illustration 1: Entry into Monopolized Industry (Cont.)

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  - The cost is given by
    - $C_1(q_1) = cq_1$,
    - $C_2(q_2) = cq_2 + f$.
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  - The profits of the firms are
    - $\pi_1((Out, q_1)) = q_1(\alpha - P(q_1)) - cq_1$, and

Illustration 1: Entry into Monopolized Industry (Cont.)

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  - The profits of the firms are
    - $\pi_1((Out, q_1)) = q_1(\alpha - P(q_1)) - cq_1$, and
    - $\pi_2((Out, q_1)) = 0$,

Illustration 1: Entry into Monopolized Industry (Cont.)

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    - $\pi_2((Out, q_1)) = 0$,
    - $\pi_1((In, (q_1, q_2))) = q_1(\alpha - P(Q)) - cq_1$,

Illustration 1: Entry into Monopolized Industry (cont.)

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$$P(Q) = \begin{cases} \alpha - Q & Q < \alpha \\ 0 & Q \geq \alpha. \end{cases}$$

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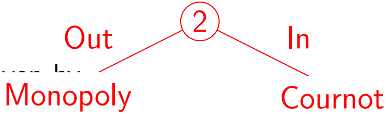
  - The fixed cost to enter for firm 2 is f.
  - The profits of the firms are
    - $\pi_1((Out, q_1)) = q_1(\alpha - P(q_1)) - cq_1$, and
    - $\pi_2((Out, q_1)) = 0$,
    - $\pi_1((In, (q_1, q_2))) = q_1(\alpha - P(Q)) - cq_1$,
    - $\pi_2((In, (q_1, q_2))) = q_2(\alpha - P(Q)) - cq_2 - f$.

Illustration 1: Entry into Monopolized Industry (Cont.)



- **Preferences:**
  - The inverse demand function is given by

$$P(Q) = \begin{cases} \alpha - Q & Q < \alpha \\ 0 & Q \geq \alpha. \end{cases}$$

  - The cost is given by
    - $C_1(q_1) = cq_1$,
    - $C_2(q_2) = cq_2 + f$.
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Illustration 1 Solution

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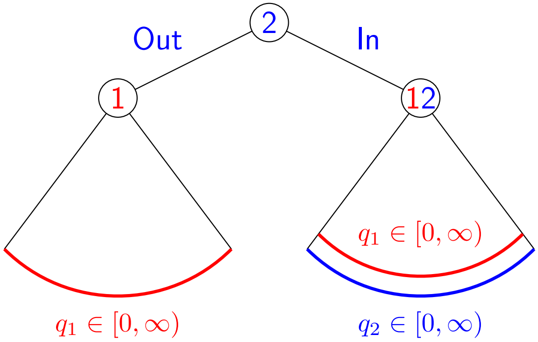
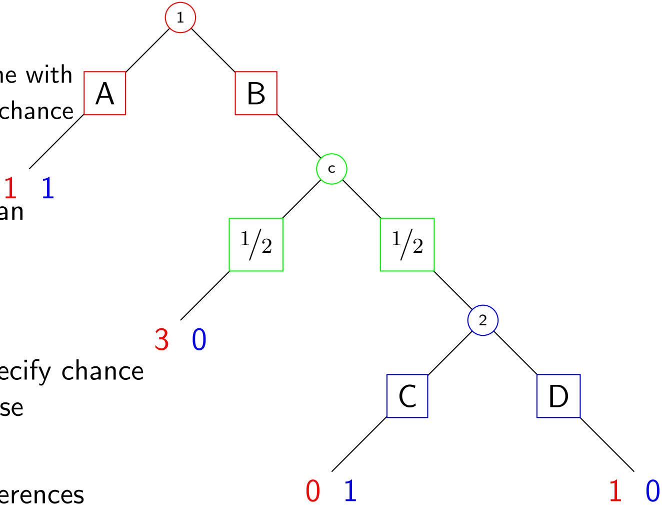


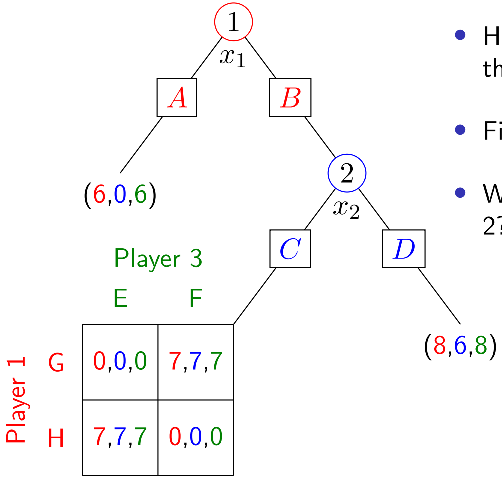
Exogenous Uncertainty

- Here is an extensive game with perfect information and chance moves.
    - Player function can assign "chance" to some histories.
    - A player must specify chance probability at these histories.
    - Players have preferences over these lotteries.



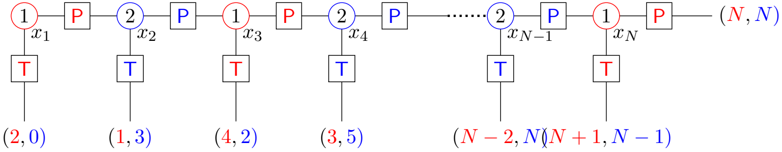
- What is the SPNE?

Discussion (Example 1)



- How many non-terminal histories are there?
- Find all SPNE.
- What would you do if you were player 2?

Discussion: Centipede Game (Example 2)



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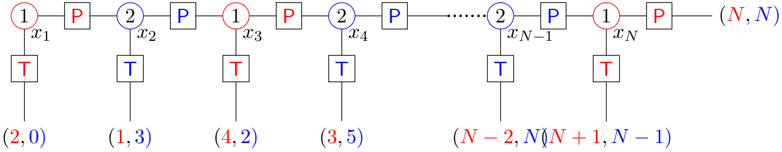

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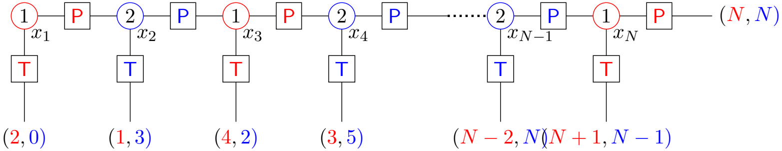
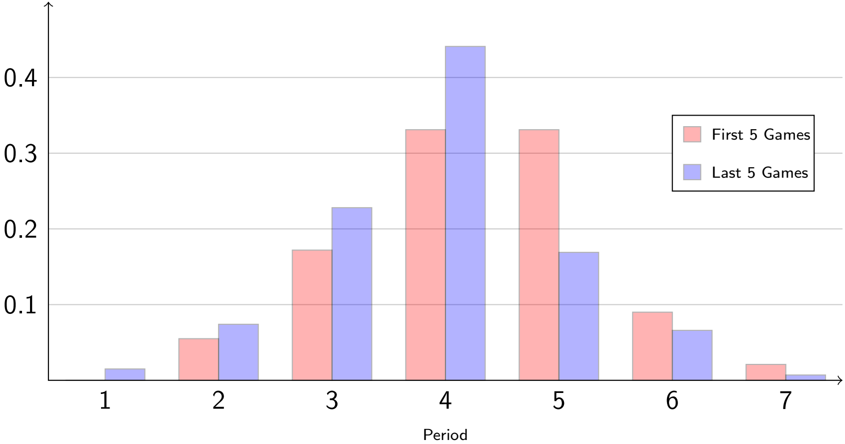
Discussion: Centipede Game (Example 2)



- What is the SPNE?
- What would you do if you were player 1?

Discussion: Centipede Game (Example 2) (Cont.)

- McKelvey and Palfrey (1992)



Discussion: Chain Store Game (Example 3)

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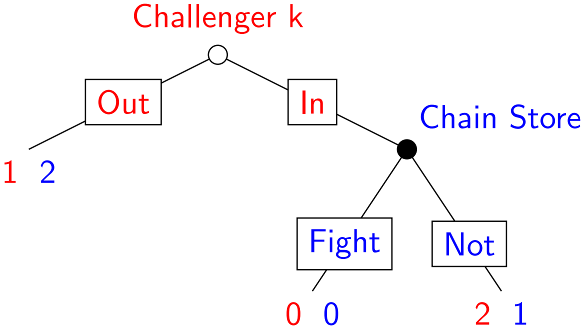
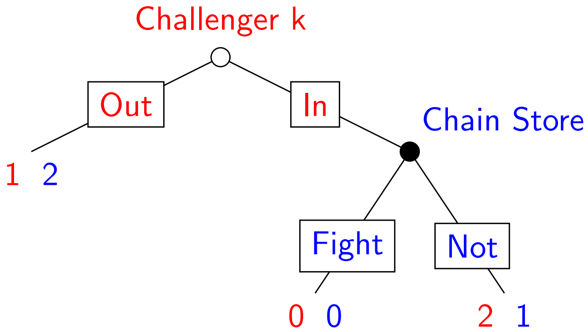
$$\pi_{Challengerk} = \begin{cases} 0 & e_k = (In, F) \\ 1 & e_k = (Out) \\ 2 & e_k = (In, N). \end{cases}$$

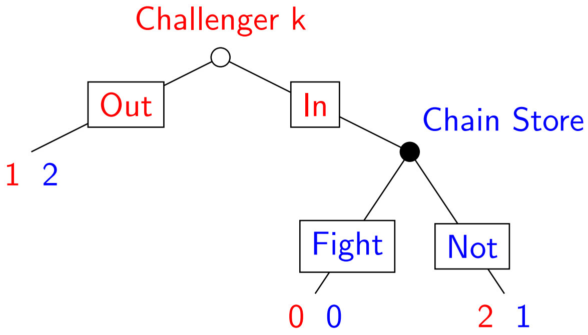
Discussion: Chain Store Game (Example 3) (Cont.)



Discussion: Chain Store Game (Example 3) (Cont.)



Discussion: Chain Store Game (Example 3) (Cont.)



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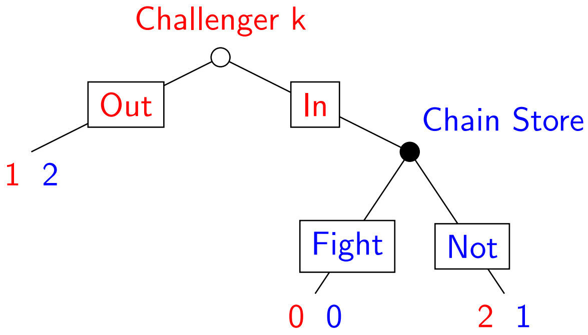

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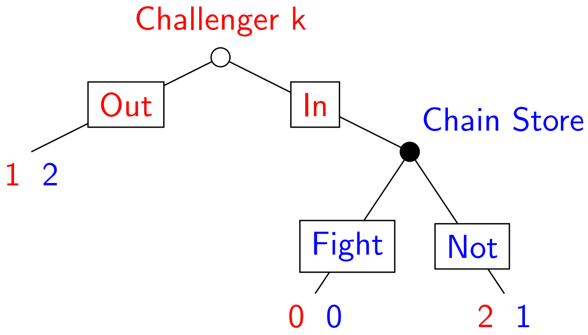

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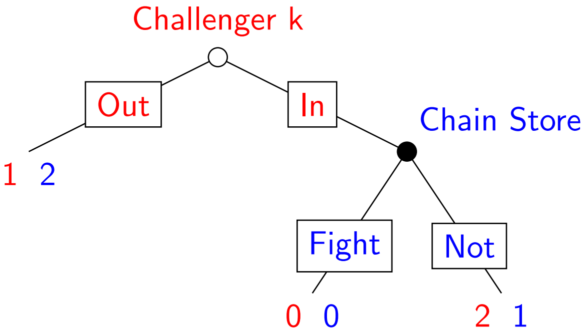

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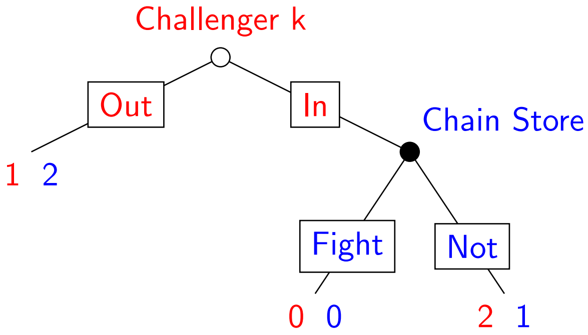

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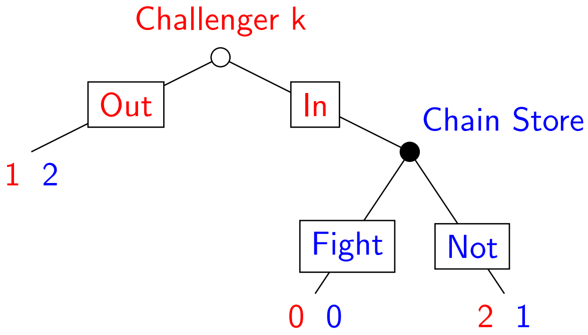

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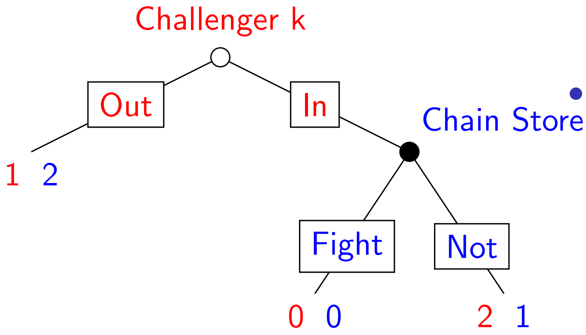

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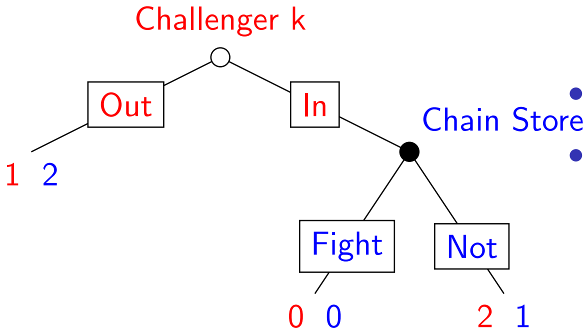
Discussion: Chain Store Game (Example 3) (Cont.)



- How many non-terminal histories are there?
- What is the SPNE?
- What would you do if you were the chain store?