Extensive Games with Perfect Information: Theory

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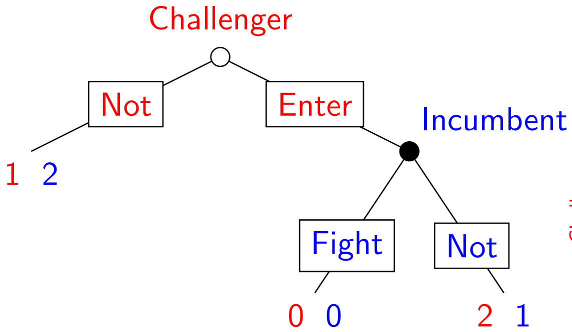
Challenger \ Incumbent	Fight	Not
Enter	0,0	2,1
Not	1,2	1,2

Game Tree



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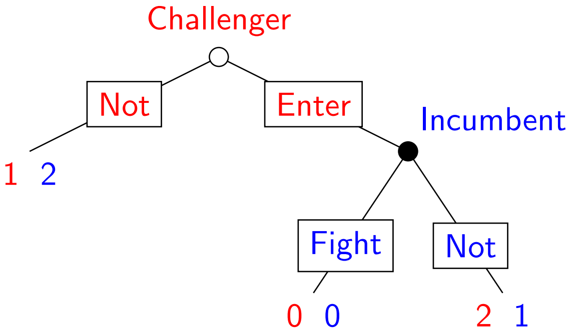

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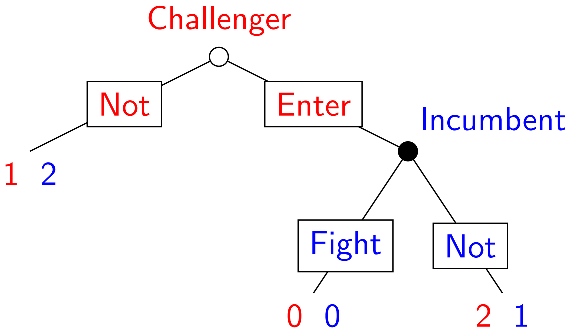

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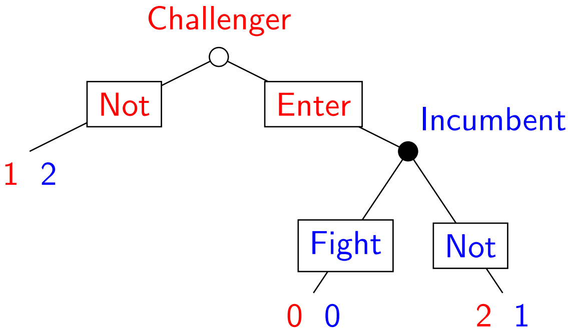

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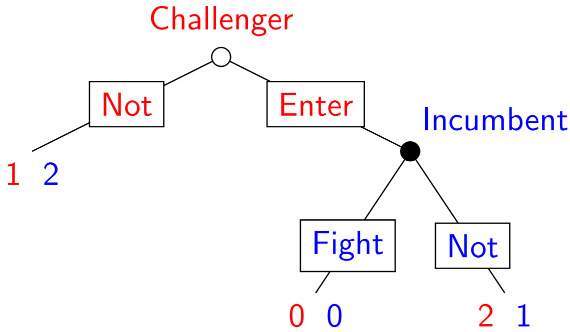

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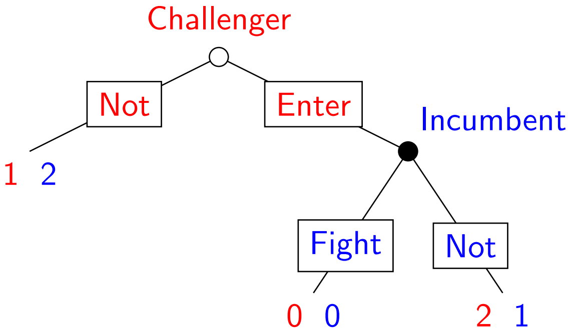
- **Terminal Histories:**

Game Tree



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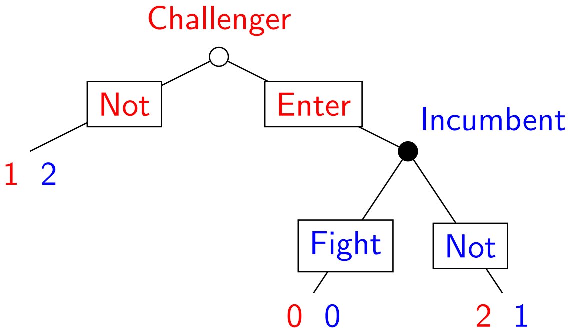

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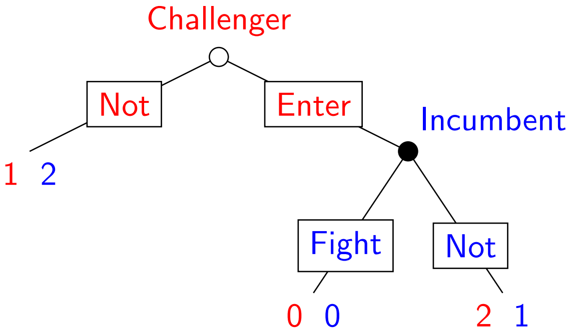

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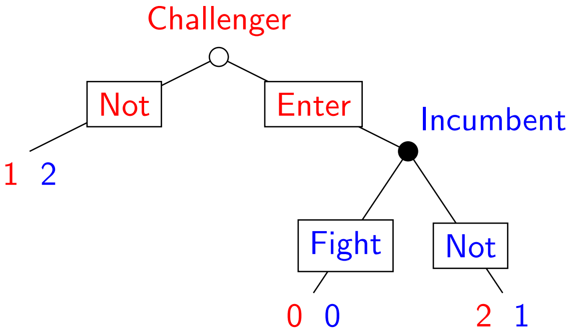

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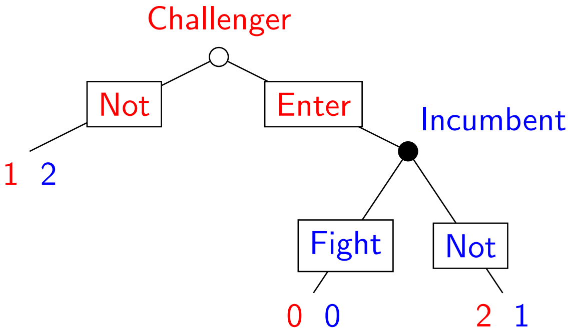

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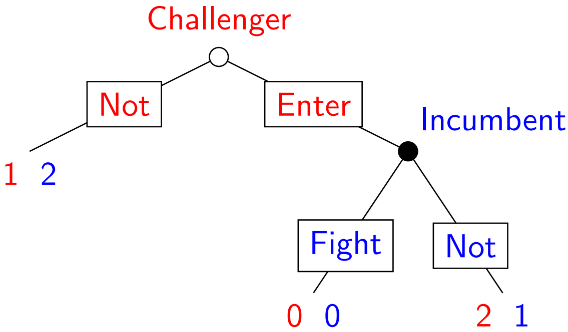

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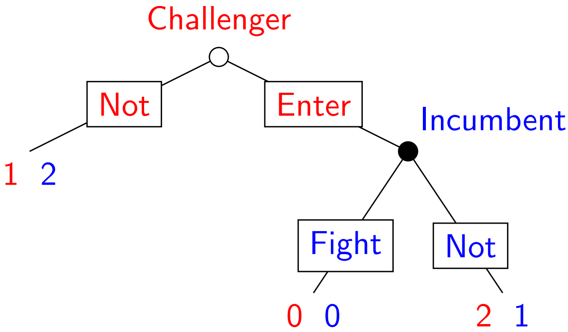

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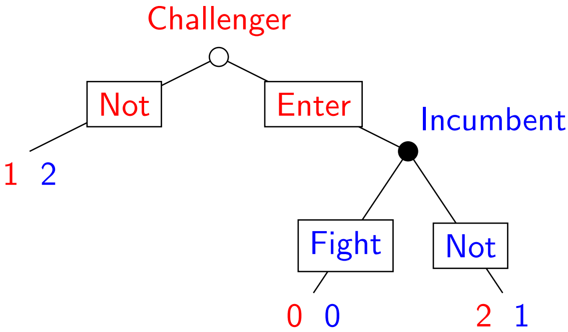

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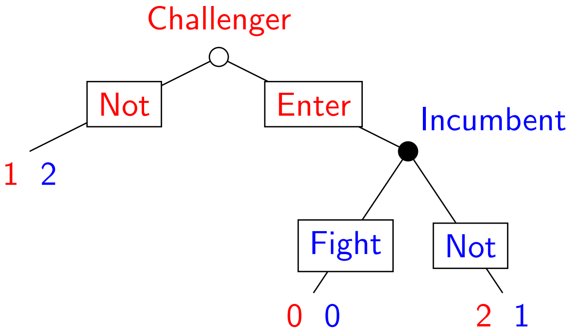

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- **Preferences:**
  - On the tree.

Extensive Game with Perfect Information

Definition

An **extensive game with perfect information** consists of:

- a set of **players**,
- a set of sequences (**terminal histories**) with the property that no sequence is a proper subhistory of any other sequence,
- a function (the **player function**) that assigns a player to every sequence that is a proper subhistory of some terminal history, and
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- Actions are not specified but can be inferred from terminal histories; that is, $A(h) = \{a | (h, a) \text{ is a history}\}$.

Strategies

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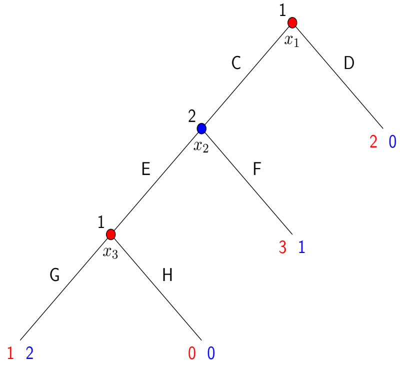
Definition

A strategy profile s^* in an extensive form game with perfect information is a **Nash equilibrium** if for all players i,

$$u_i\left(s_i^*, s_{-i}^*\right) \geq u_i\left(s_i, s_{-i}^*\right) \text{ for all strategies } s_i.$$

Strategies (Example)



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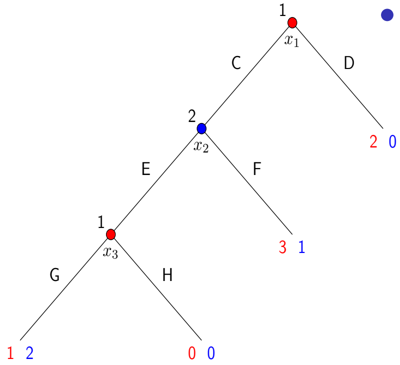

- Player 1's strategies are:

#		
1		
2		
3		
4		

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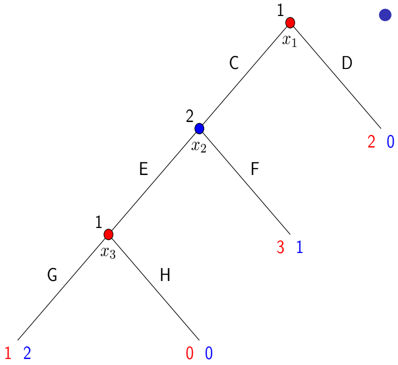

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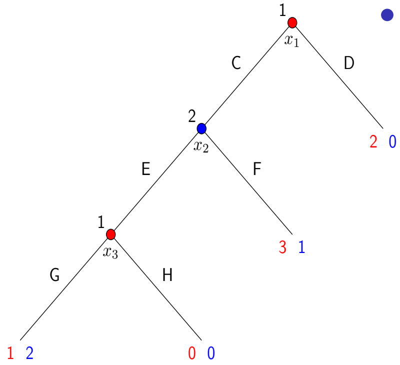

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1	C	G
2		
3		
4		

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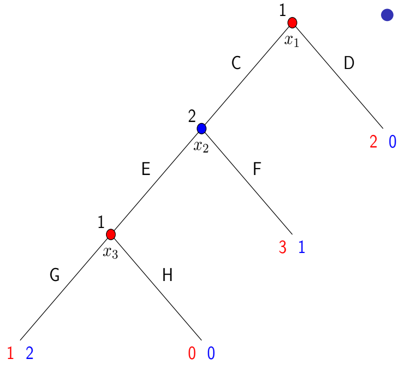

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#	Choice at x_1	Choice at x_3
1	C	G
2	C	H
3		
4		

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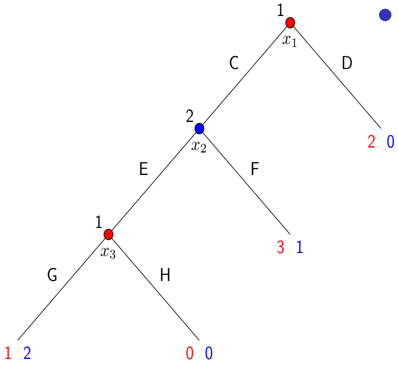

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1	C	G
2	C	H
3	D	G
4		

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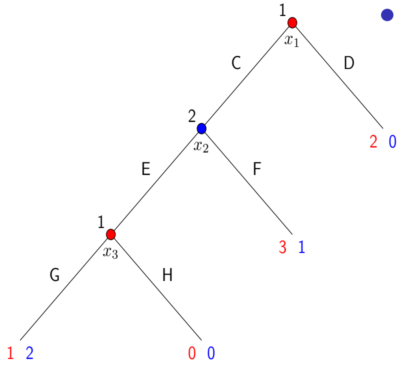

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4	D	H

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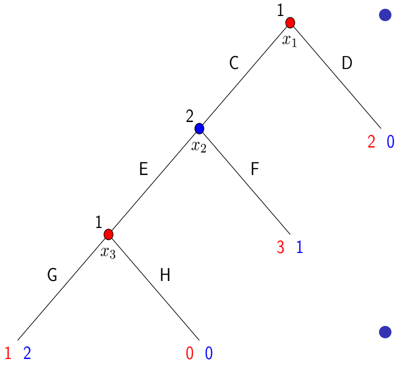

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#	Choice at x_1	Choice at x_3
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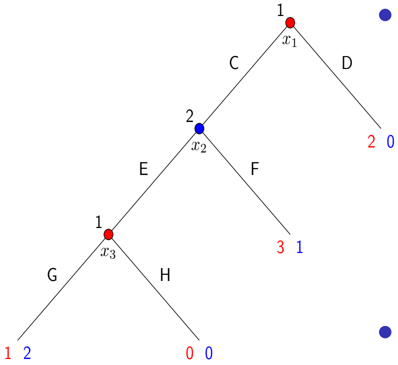
2	C	H
3	D	G
4	D	H

- Player 2's strategies are:

#	
1	
2	

Strategies (Example)



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1	
2	

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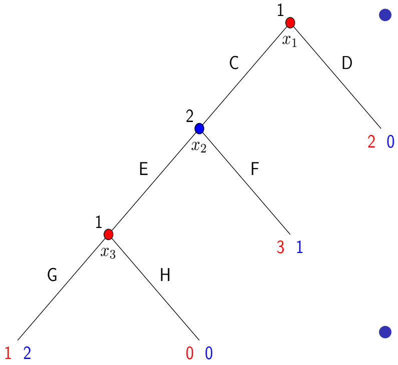

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1	C	G
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3	D	G
4	D	H

- Player 2's strategies are:

#	Choice at x_2
1	E
2	

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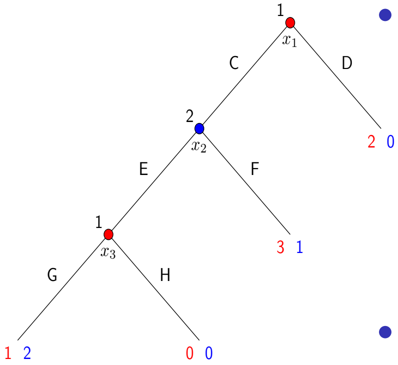

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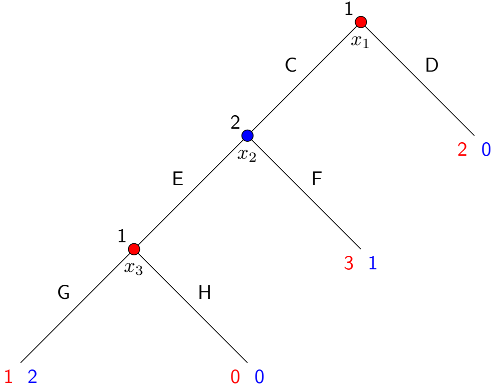
1	E
2	F

Finding Nash Equilibria (Example)



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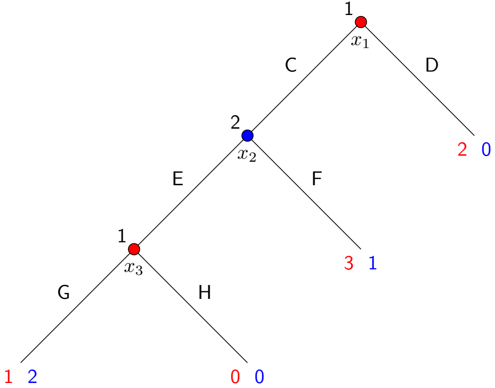

	E	F
CG		
CH		
DG		
DH		

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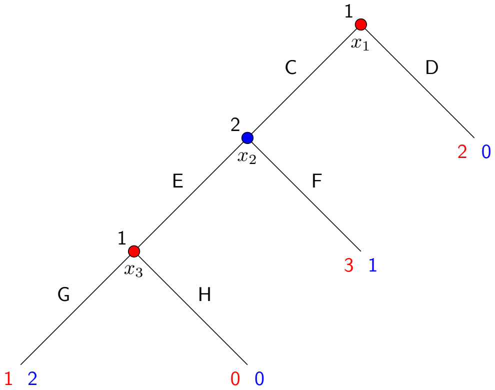

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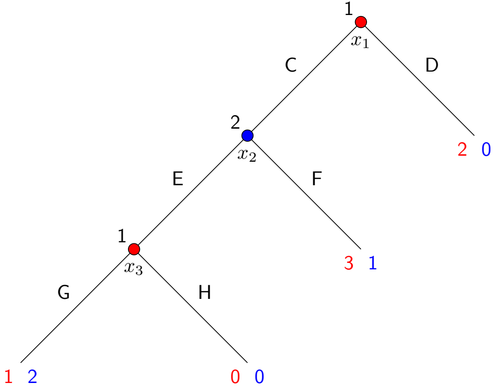

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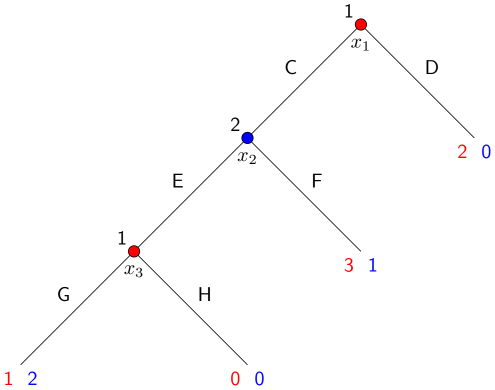

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CH	0 0	
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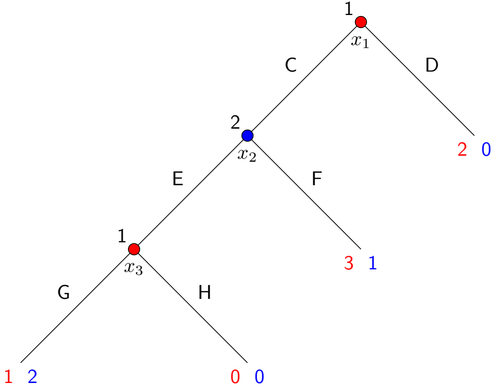

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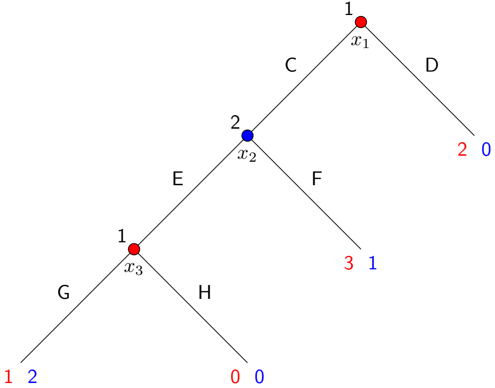

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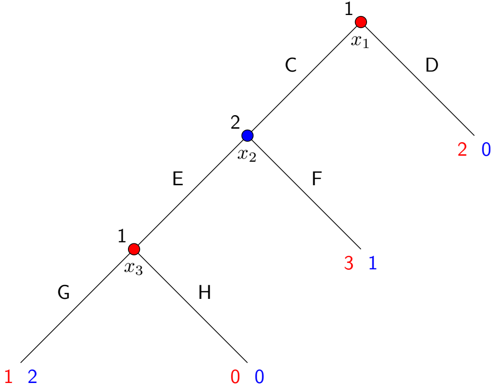

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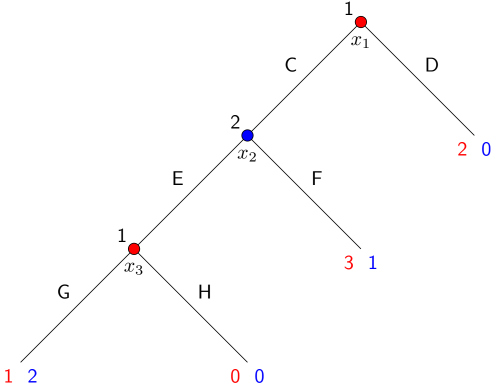

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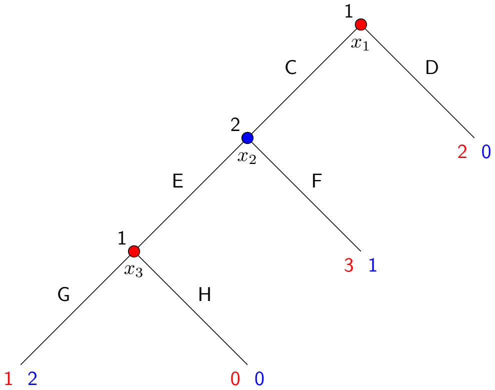

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Subgame

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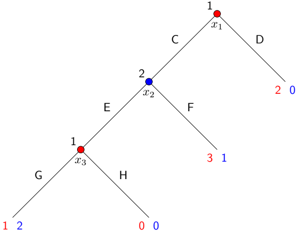
Let Γ be an extensive game with perfect information, with player function P. For any nonterminal history h of Γ, the **subgame** $\Gamma(h)$ following the history h is the following extensive game.

Subgame



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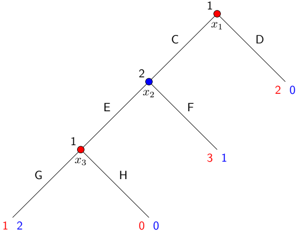

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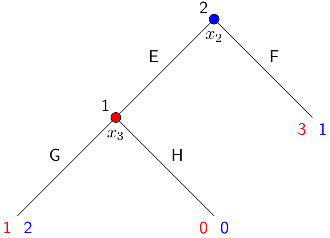

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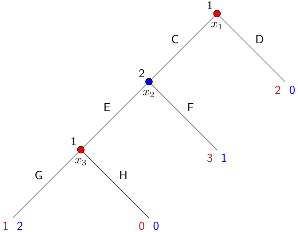

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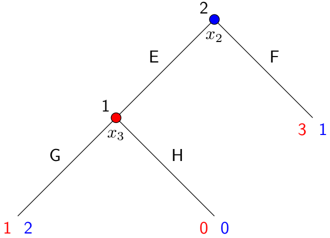

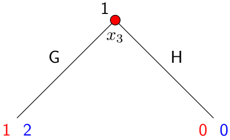

Subgame Perfect Nash Equilibrium

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Every Subgame Perfect Nash equilibrium is a Nash equilibrium.

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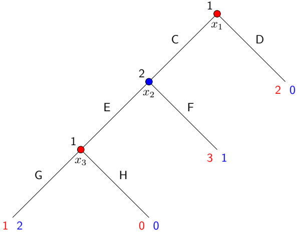

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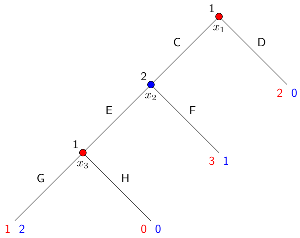

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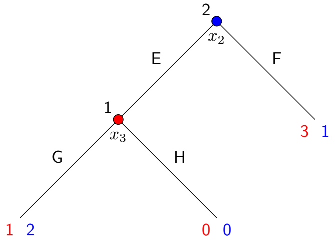

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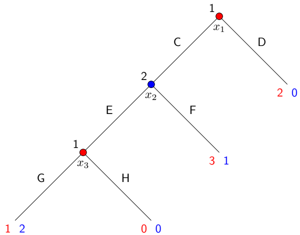
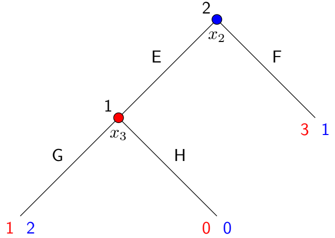
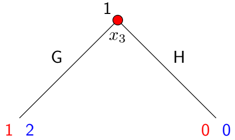


Definition

A strategy profile s^* in an extensive form game with perfect information is a **Subgame Perfect Nash Equilibrium** (SPNE) if the strategy s^* is a Nash equilibrium for every subgame.





Every Subgame Perfect Nash equilibrium is a Nash equilibrium.

Subgame Perfect Nash Equilibrium (Example)

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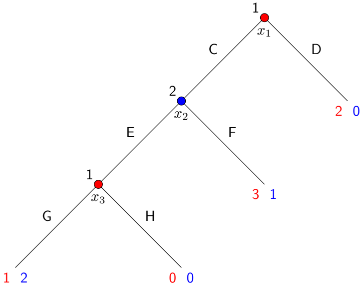
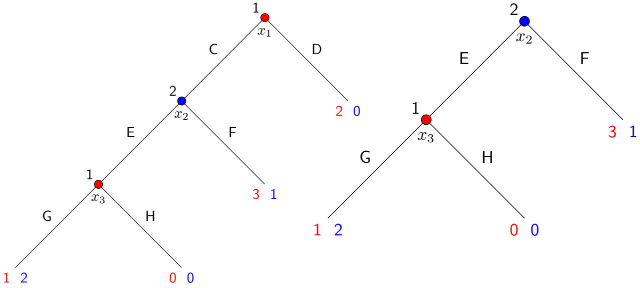


Subgame Perfect Nash Equilibrium (Example)



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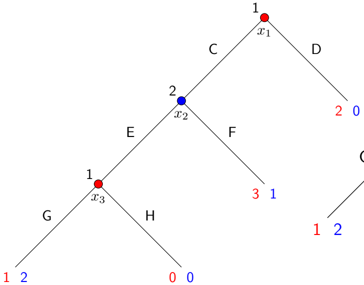
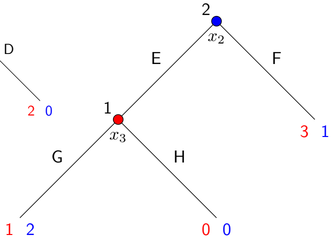
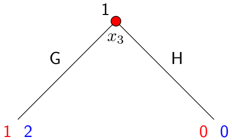
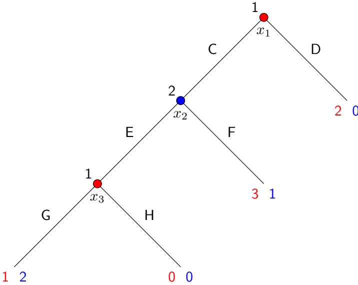






Backward Induction

- Backward induction works as follows.
  - One starts at the very last subgame;
  - in that last subgame, one finds the equilibrium;
  - the subgame is, then, replaced with the respective equilibrium payoffs;
  - the process continues in the penultimate subgame and so on and so forth until you reach the very first subgame.



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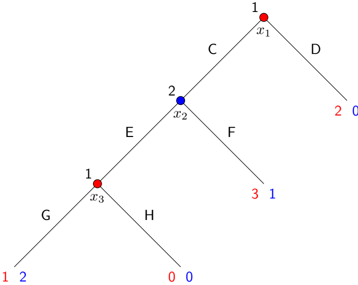
- A SPNE always exists.



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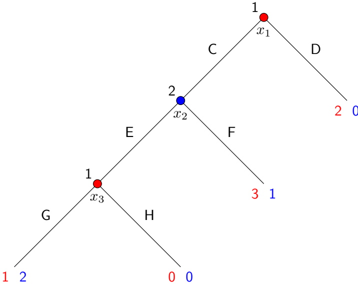
- A SPNE always exists.
- Backward induction always provides



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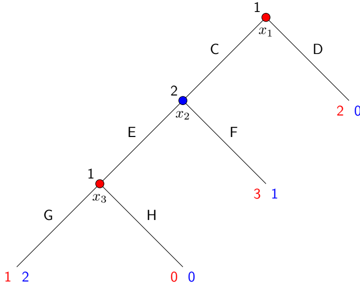
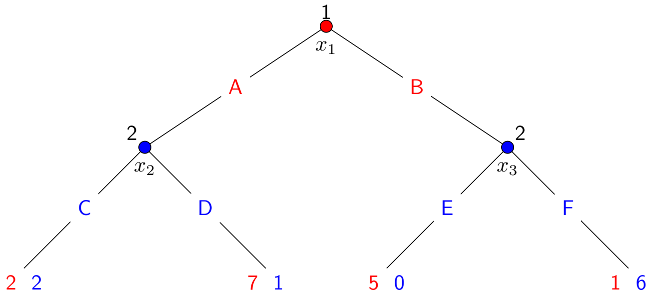
- A SPNE always exists.
- Backward induction always provides all SPNE.

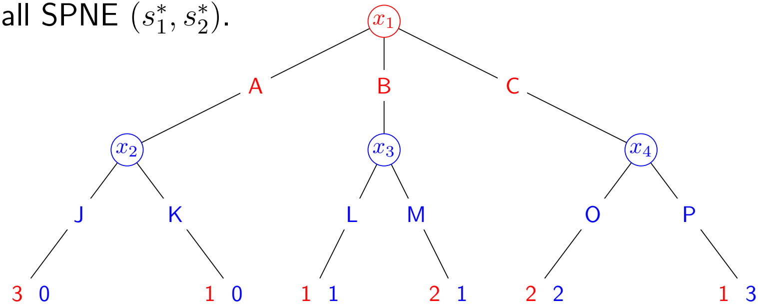


Backward Induction (Cont.)



Exercise on Finding all SPNE

- Find all SPNE (s_1^*, s_2^*).



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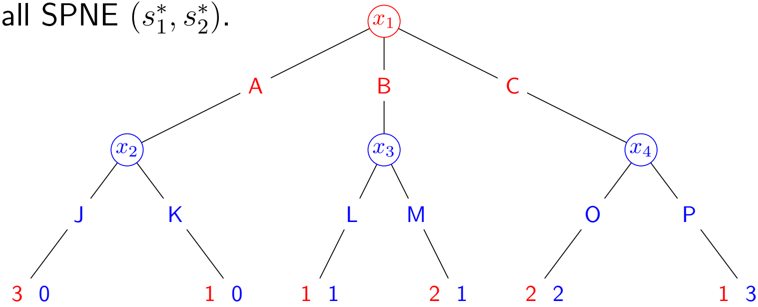

$$(A, JLP)$$

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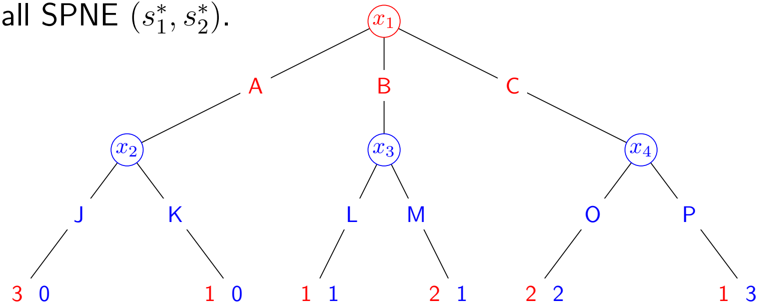

$$(A, JLP) \quad (A, KLP)$$

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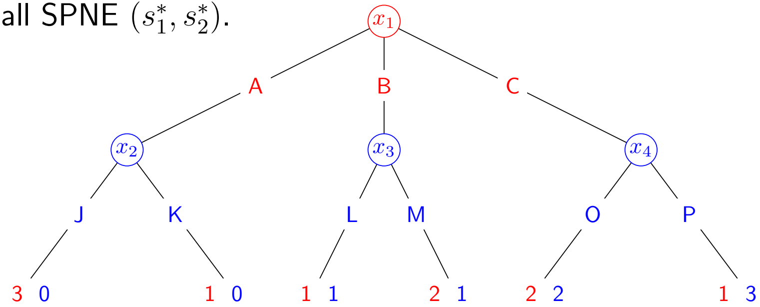

$$(A, JLP) \quad (A, KLP) \quad (B, KLP)$$

Exercise on Finding all SPNE

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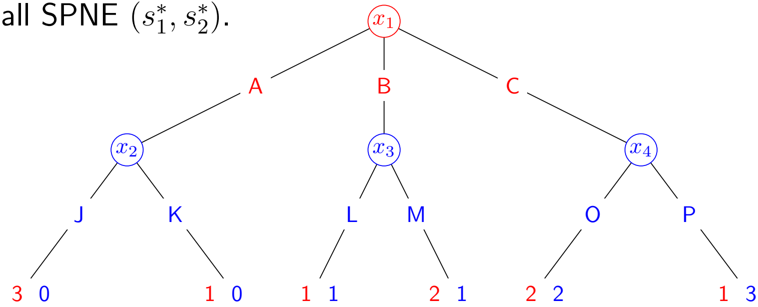

$$(A, JLP) \quad (A, KLP) \quad (B, KLP)$$

$$(C, KLP)$$

Exercise on Finding all SPNE

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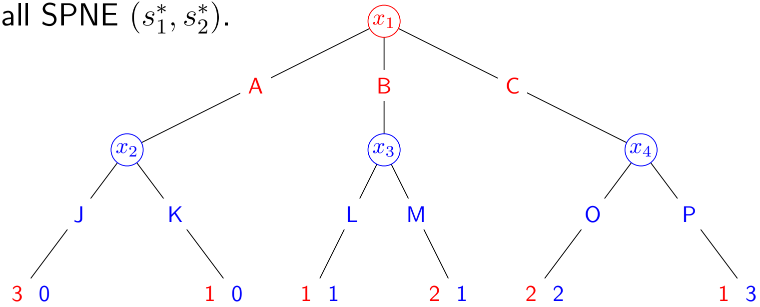
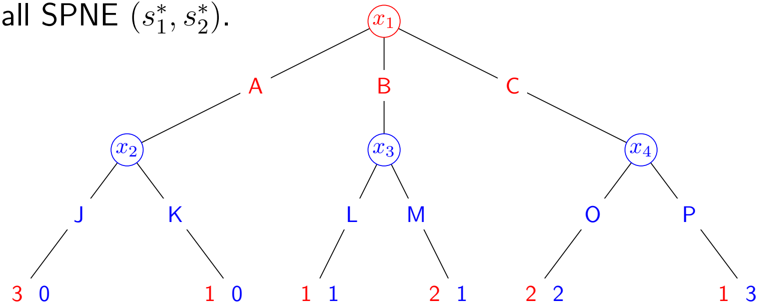


$$(A, JLP) \quad (A, KLP) \quad (B, KLP)$$

$$(C, KLP) \quad (A, JMP)$$

Exercise on Finding all SPNE

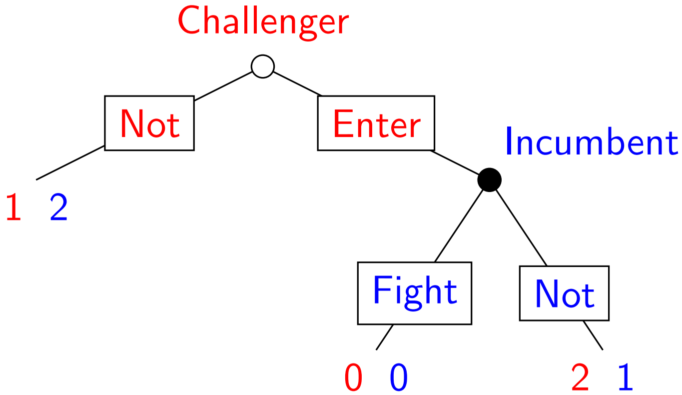
- Find all SPNE (s_1^*, s_2^*).



$$(A, JLP) \quad (A, KLP) \quad (B, KLP)$$

$$(C, KLP) \quad (A, JMP) \quad (B, KMP)$$

Recall our Motivational Example



All-Pay Auction (Exercise)

- The auction is for a \$2 bill.
- There are two bidders that alternate their bids.
- Bids must be integers and higher than the previous bid.
- A player can pass in which case the other player wins.
- Both players pay their final bids.
- Assume each player only has \$3, so that the only possible bids are \$1, \$2, and \$3.