Mixed Strategy Equilibrium

Matching Pennies

Player 2

Player 1		Heads	Tails
	Heads	1, −1	−1, 1
	Tails	−1, 1	1, −1

- The ordinal preferences for player 1 are $(H,H) \sim_1 (T,T) \succ_1 (H,T) \sim_1 (T,H)$.
- SGWOP has no (pure) Nash Equilibrium.
- A lottery is a probability distribution over outcomes.
- To talk about steady state in this game, we need to define preferences over lotteries.

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$$L \succeq M \iff pu(A)+(1-p)\,u(B) \geq pu(C)+(1-p)\,u(D).$$

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Mixed Strategy Nash Equilibrium

Definition

The mixed strategy profile α^* in a strategic game with vNM preferences is a **mixed strategy Nash equilibrium** if for every player i,

$$U_i\left(\alpha_i^*, \alpha_{-i}^*\right) \geq U_i\left(\alpha_i, \alpha_{-i}^*\right) \text{ for all } \alpha_i \text{ of player } i,$$

where $U_i(\alpha)$ is player i's expected payoff to the mixed strategy α.

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- Recall, we denote the best response function as $B_i\left(\alpha_{-i}\right)$.
- A profile α^* is a Nash equilibrium if $\alpha_i^* \in B_i\left(\alpha_{-i}^*\right)$ for all i.

Mixed Strategy Nash Equilibrium (Cont.)

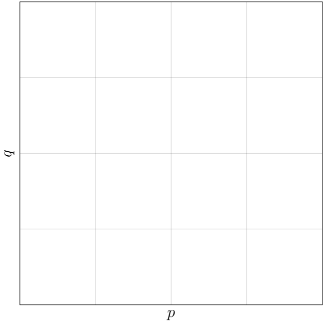
Proposition (Nash (1949))

Every SGWvNMP in which each player has finitely many actions has a mixed strategy Nash equilibrium.

Exercise 1 on Mixed Strategy Nash Equilibrium

- Find all mixed strategy Nash equilibria.

	L	R
U	3,7	6,4
D	5,1	2,8



Exercise 2 on Mixed Strategy Nash Equilibrium

- Find all mixed strategy Nash equilibria.

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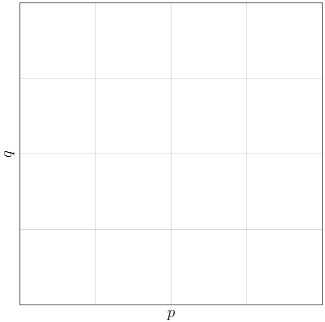

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  $$U_i\left(\alpha_i^*, \alpha_{-i}^*\right) \geq U_i\left(a_i, \alpha_{-i}^*\right).$$
- Why? A player would not want to randomly choose between two actions if one always leads to a higher payoff.

Properties (Cont.)

- Each player's expected payoff in equilibrium is her expected payoff to any of her actions that she uses with positive probability.

	L [0]	C [$\frac{1}{3}$]	R [$\frac{2}{3}$]
T [$\frac{3}{4}$]	·,2	3,3	1,1
M [0]	·,·	0,·	2,·
B [$\frac{1}{4}$]	·,4	5,1	0,7

Exp. Payoff of 1: $\frac{5}{3}$; Exp. Payoff of 2: $\frac{5}{2}$.

Symmetric Games

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A two-player SGWvNMP is **symmetric** if the players' set of actions are the same and the players' preferences are represented by payoff function u_1 and u_2 for which $u_1(a_1, a_2) = u_2(a_2, a_1)$ for every action pair (a_1, a_2).

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A mixed strategy profile α^* in a symmetric SGWvNMP is a **symmetric Nash equilibrium** if it is a Nash equilibrium and α_i^* is the same for every player i.

- A symmetric game will always have a symmetric Nash equilibrium.

Exercise on Reporting a Crime

- The game consists of the following elements.
  - **Players**: There are n bystanders.
  - **Actions**: Can either C (call) or D (do not call).
  - **Payoffs**:

$$u_i\left(C, \text{Anything}\right) = v - c > 0$$
$$u_i\left(D, \text{At least one call}\right) = v$$
$$u_i\left(D, \text{No calls}\right) = 0$$

- Questions:
  1. Are there any pure strategy symmetric Nash equilibria?
  2. Can you find all symmetric equilibria?

Strictly Dominated Strategy

Definition

In a SGWvNMP, player i's mixed strategy α_i strictly dominates her actions a_i' if

$$u_i(\alpha_i, a_{-i}) > u_i(a_i', a_{-i}) \text{ for every list } a_{-i} \text{ of the other players' actions,}$$

where u_i is player i's payoff function. We say that the action a_i' is **strictly dominated**.

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- L is not strictly dominated by either M or R.
- L is strictly dominated by $1/2 M + 1/2 R$.
- A strictly dominated strategy will never be played with positive probability in a Nash equilibrium.

Weakly Dominated Strategy

Definition

In a SGWvNMP, player i's mixed strategy α_i weakly dominates her actions a_i' if

$u_i(\alpha_i, a_{-i}) \geq u_i(a_i', a_{-i})$ for every list a_{-i} of the other players' actions,

and,

$u_i(\alpha_i, a_{-i}) > u_i(a_i', a_{-i})$ for at least one list a_{-i} of the other players' actions,

where u_i is player i's payoff function. We say that the action a_i' is **weakly dominated**.

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- Pure strategy NE survive when randomization is allowed.