Pattern Mining

Objective

The best known results in the theory of repeated games, the folk theorems, focus attention on the multiplicity of equilibria in such games, a source of great consternation for some. We consider multiple equilibria a virtue – how else can one hope to explain the richness of behavior that we observe around us?

—Mailath and Samuelson (2006)

- Uncover new regularities in observed play.
- Repeated games are an obvious candidate for pattern mining.
- Are there observable patterns of play?
- We conducted experiments on a rich set of repeated games with a high discount factor.
- We developed two pattern mining approaches.
  - Action convergence
  - k-means clustering

Stage Games

	A	B
A	3,3	1,4
B	4,1	2,2

(a) Prisoner's Dilemma

	A	B
A	1,1	4,2
B	2,4	1,1

(b) Battle of the Sexes

	A	B
A	3,3	0,2
B	2,0	1,1

(c) Stag Hunt

	A	B
A	3,3	1,4
B	4,1	0,0

(d) Chicken

	A	B
A	3,3	1,2
B	2,1	0,0

(e) Common Interest

	A	B
A	2,2	2,2
B	5,3	3,5

(f) Samaritan's Dilemma

	A	B
A	2,2	2,2
B	3,1	0,0

(g) Ultimatum

	A	B
A	4,1	1,2
B	1,2	2,1

(h) Unique Mixed

What is a Pattern?

- A **pattern of play** of length $n < \infty$ is a sequence $p \in \mathcal{A}^n$, written as $p = (p^1 p^2 \dots p^n)$, such that there exists no $0 < \ell < n$ such that $p^t = p^{\mathrm{mod}_n(t+\ell)}$ for all $t = 1, \dots, n$.

A
A

A	A
A	A

B	A
A	B

B	A	B	A
A	B	A	B

- Let $|p|$ denote the length of pattern p.

- We work with equivalence classes of patterns

$$[p] := \left\{ q \in \mathcal{A}^{|p|} : \text{there exists some } 0 < \ell < n \text{ such that } p^k = q^{\mathrm{mod}_{|p|}(k+\ell)} \text{ for all } 1 \leq k \leq |p| \right\}.$$

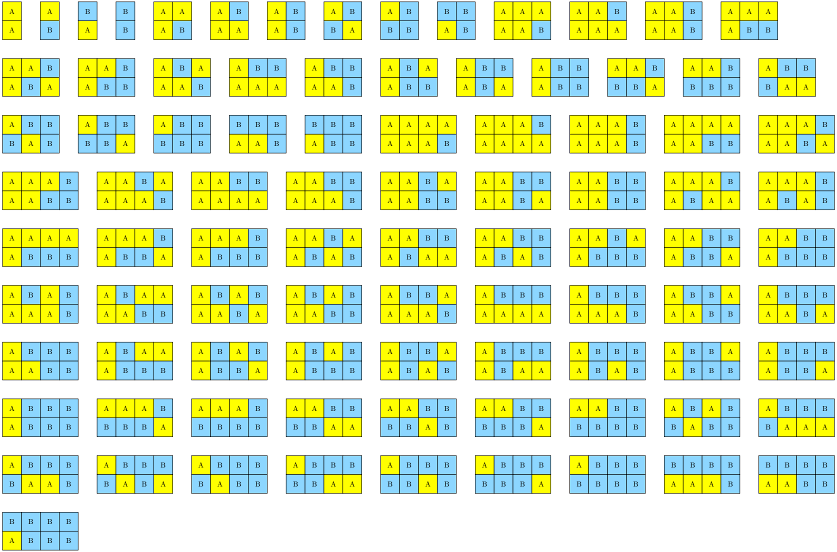
A	B	A
A	A	B

A	A	B
B	A	A

B	A	A
A	B	A

- There are a total of 964 pattern classes in $\mathcal{P}$, indexed as $\mathcal{P} = \{p_k\}_{k=1}^{964}$. We allow for patterns of length ≤ 6.

Patterns of Length ≤ 4



Match Function

- Given a sequence of observed action profiles, $s \in \bigcup_{n=1}^{\infty} \mathcal{A}^n$, a pattern p **matches** s in period t if for some $q \in [p]$, $s^{t+k-1} = q^k$ for all $1 \leq k \leq |p|$.

- If $|p| > |s| - t + 1$, then the match is undefined, because there are not enough remaining periods in the sequence to fully evaluate the pattern.

- If the match is not undefined and yet the pattern is not a match, then this is a mismatch or an error.

- Let
$$m(p, s, t) = \begin{cases} 1 & \text{if } p \text{ matches } s \text{ in period } t \text{ or the match is undefined} \\ 0 & \text{otherwise} \end{cases}$$
be the match function between p and s in t.

Examples

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B
	B	A	B	A	A	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B	B	B	A
Pattern #1	1	X	1	X	X	X	1	X	1	X	1	X	1	X	1	X	1	X	1	X	1	X	1	X	1	X
Pattern #2	1	2	1	X	X	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	X	X	1	-
Pattern #3	X	3	X	X	X	3	X	3	X	3	X	3	X	3	X	3	X	3	X	3	X	X	X	X	-	-
Pattern #4	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	3	-	-
Pattern #5	3	4	5	6	1	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	-	-	-	-	-

Pattern #1

A
B

Pattern #2

A	B
B	A

Pattern #3

A	B	B
B	A	A

Pattern #4

A	B	B
B	A	B

Pattern #5

A	B	A	B	A	B
A	A	B	A	B	A

Action Convergence

- Our first technique is inspired from the string-searching literature (Boyer and Moore (1977), Knuth, Morris, and Pratt (1977)).

- For any p, s and interval of periods T, let

$$\mathcal{X}(p,s,T) = \sum_{t \in T} (1 - m(p,s,t))$$

  be the number of periods in T in which pattern p mismatches sequence s.

- A sequence s is said to x-**converge to pattern** p over T if $min_{p \in \mathcal{P}} \mathcal{X}(p,s,T) \leq x$ and $argmin_{p \in \mathcal{P}} \mathcal{X}(p,s,T) = p$. If there is no such p, the sequence is x-**divergent**.

- We focus on 2-convergence and choose T to cover the first or the last 20 periods.

k-Means

- For each pattern $p \in \mathcal{P}$, we compute the proportion of periods at which p matches s. Specifically, let $\phi > 0$ be a discount term and let the weighted frequency of pattern p against s be defined as

$$f(s, p, \phi) = \frac{\sum_{t=1}^{|s|-|p|+1} \phi^t m(p, s, t)}{\sum_{t=1}^{|s|-|p|+1} \phi^t},$$

- and for every $i = 1, \ldots, d$, let

$$v_i(s) = \begin{cases} f(s, p_i, \delta) & i \leq 964 \\ f(s, p_{i-964}, 1) & 964 < i \leq 1,928 \\ f\left(s, p_{i-1928}, \delta^{-1}\right) & 1,928 < i \end{cases}$$

  be the i^{th} entry of the attribute vector for sequence s.

- We capture the *evolution of play*. This yields a vector with $d = 3 \times 964 = 2,892$ entries.

Experiments

Game	Acronym	# of Pairs
Prisoner's Dilemma	PD	70
Battle of the Sexes	BO	70
Stag Hunt	SH	50
Chicken	CH	70
Common Interest	CI	18
Samaritan's Dilemma	SD	60
Ultimatum	UL	60
Unique Mixed	MX	36
Total		434

- Each subject plays 3 infinitely-repeated games with a different match each time. The continuation probability is 0.99.

Action Convergence Patterns

	PD	BO	SH	CH	CI	SD	UL	MX
Data Points	70	70	50	70	18	60	60	36
Convergent	26 → 52	10 → 38	41 → 46	24 → 47	17 → 18	14 → 33	16 → 35	→ 2
A / A	20 $\underset{-1}{\to}$ 34	5 $\underset{-3}{\to}$ 4	41 $\underset{-1}{\to}$ 44	24 $\underset{-2}{\to}$ 42	17 → 18		7 $\underset{-1}{\to}$ 8	→ 1
B / B	6 $\underset{-2}{\to}$ 18		→ 2	→ 4		9 $\underset{-3}{\to}$ 13		
A / B		→ 3				2 $\underset{-2}{\to}$ 3	9 $\underset{-1}{\to}$ 24	
A B / B A		5 → 27		→ 1				
B B / A B						3 → 16		→ 1
B / A		→ 4					→ 2	
B B B B / A B B B						→ 1		
A A A A B A / A A A A A A							→ 1	
Divergent	44 → 18	60 → 32	9 → 4	46 → 23	1 →	46 → 27	44 → 25	36 → 34

Long-Run Outcomes

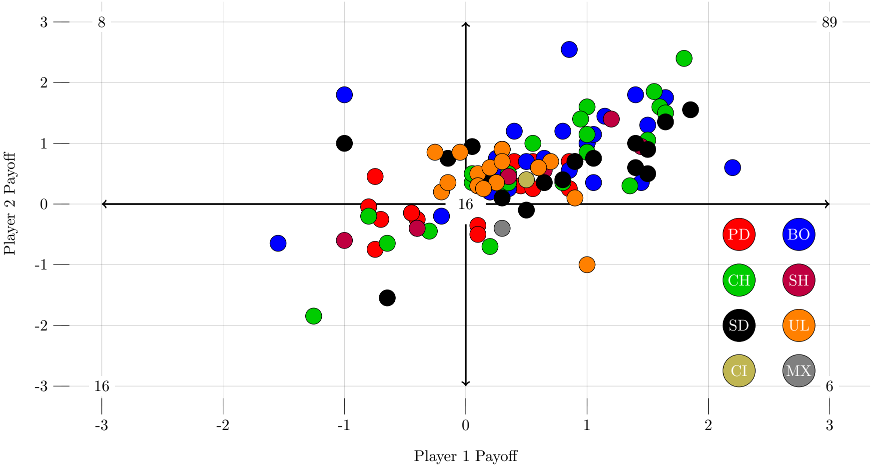
The *long-run outcomes* are the patterns of play that sequences converge to over the last 20 periods.

- **Result 1** *Convergence to a long-run outcome is predominant in the data*.
  - There are 269 (excl. MX) convergent sequences out of 398 data points (67%).
- **Result 2** *(i) Convergence is higher in the last 20 periods than in the first 20 periods. (ii) Patterns of length 2 or more are much more frequent in late rather than early convergence*.
  - There are 269 convergent sequences in the last 20 periods vs. 148 in the first 20 periods (an increase of 80%).
  - There are five times more patterns of length 2 or more amongst the sequences converging at the end than amongst those converging in the beginning.

Long-Run Outcomes (cont.)

- **Result 3** *(i) Long-run outcomes are predominantly efficient. (ii) More than half of the efficient long-run outcomes are not a static Nash equilibrium outcome. (iii) Long-run outcomes that come from divergent sequences in the first 20 periods are predominantly Pareto improving with some qualifications in the Prisoner's Dilemma.*
  - There are 237 out of 269 (88%) long-run outcomes that are Pareto efficient.
  - There are 129 out of 237 (54%) Pareto efficient long-run outcomes that are not static Nash equilibria. Thus, subjects prefer efficiency over a static Nash equilibrium.
  - There are 89 out of 135 long-run outcomes (coming from sequences that diverged in the beginning) that are strictly Pareto improving relative to the first 20 periods.
  - In the Prisoner's Dilemma, strategic dominance causes some friction.

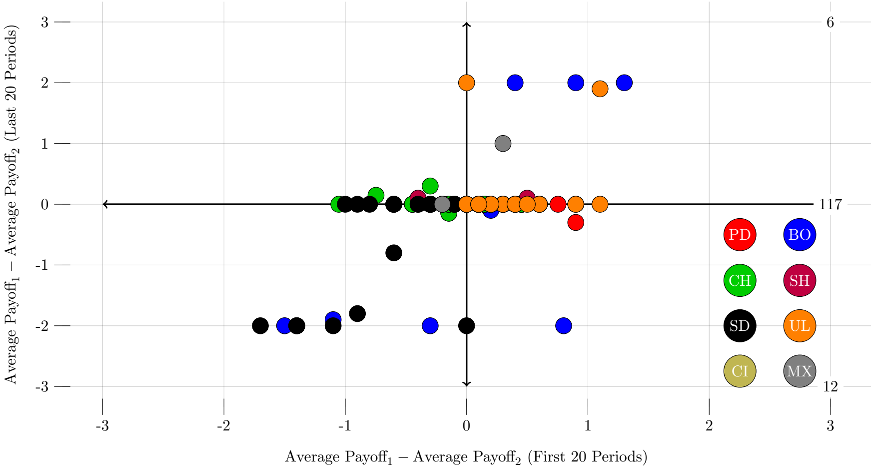
Pareto Improvements in the Long Run (Divergent → Convergent)



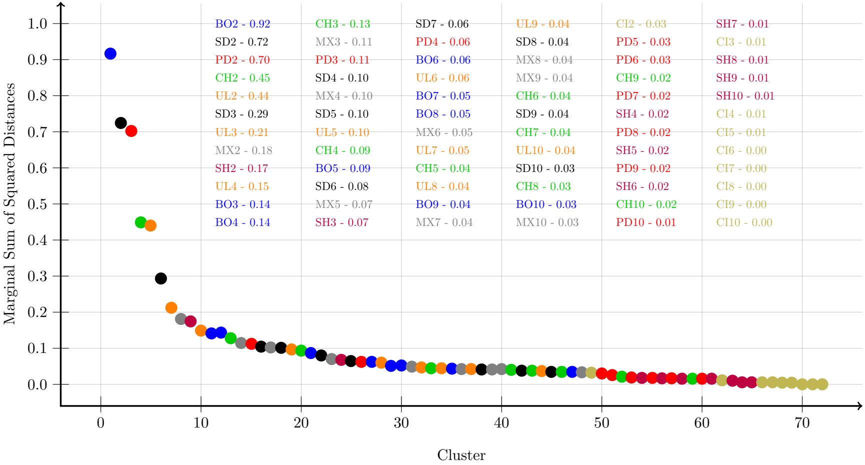
Long-Run Outcomes (cont.)

- **Result 4** *(i) Long-run outcomes are predominantly egalitarian. (ii) More than half of the egalitarian long-run outcomes are not a static Nash equilibrium. (iii) Long-run outcomes that come from divergent sequences in the first 20 periods are predominantly egalitarian with some qualifications in the Battle of the Sexes and the Samaritan's Dilemma.*
  - Excluding Unique Mixed, 245 long-run outcomes out of 269 (91%) are egalitarian.
  - Out of 245 long-run outcomes, 139 (57%) are not a static Nash equilibrium.
  - There are 117 out of 135 (88%) long-run outcomes (coming from sequences that diverged in the beginning) that are egalitarian relative to the first 20 periods.
  - The need to coordinate on alternations to arrive at the egalitarian long-run outcome seems to be problematic for some pairs that end up settling on the static Nash equilibrium.

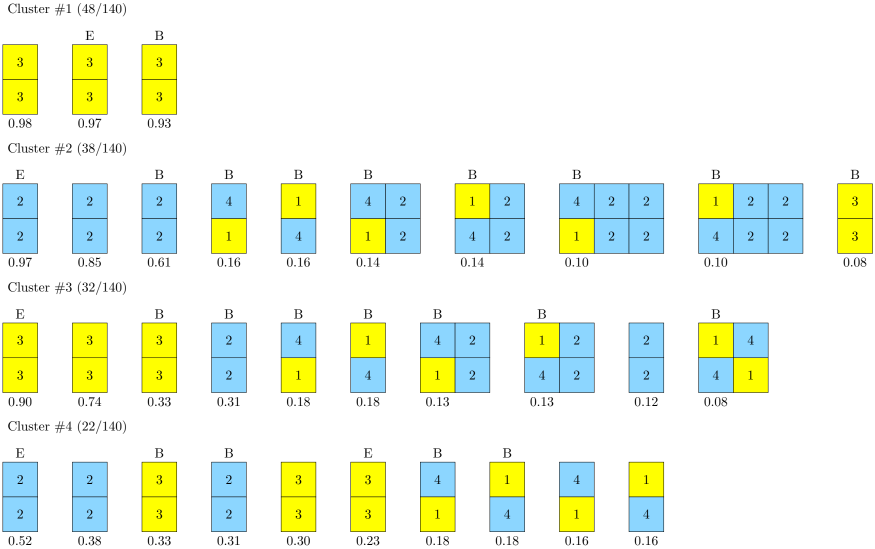
Towards Equality in the Long Run (Divergent → Convergent)



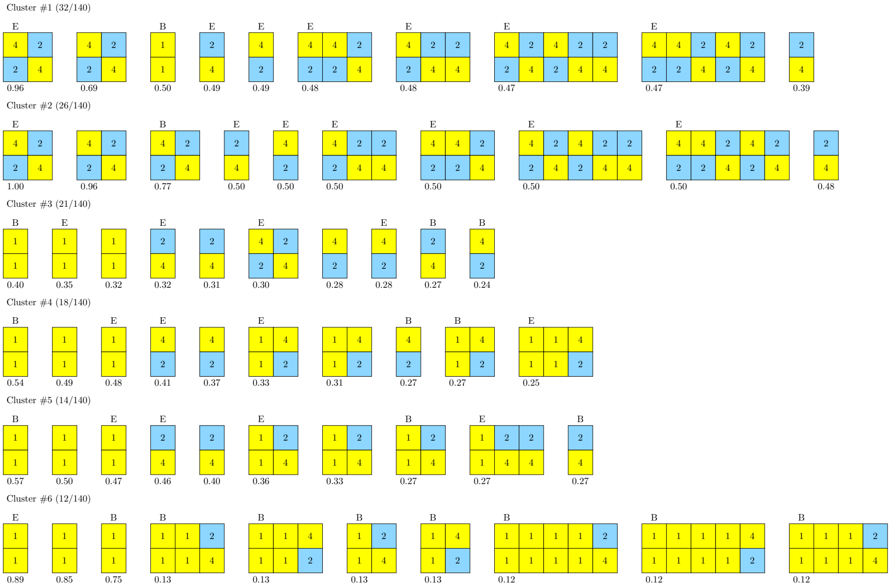
Endogenizing k in k-Means

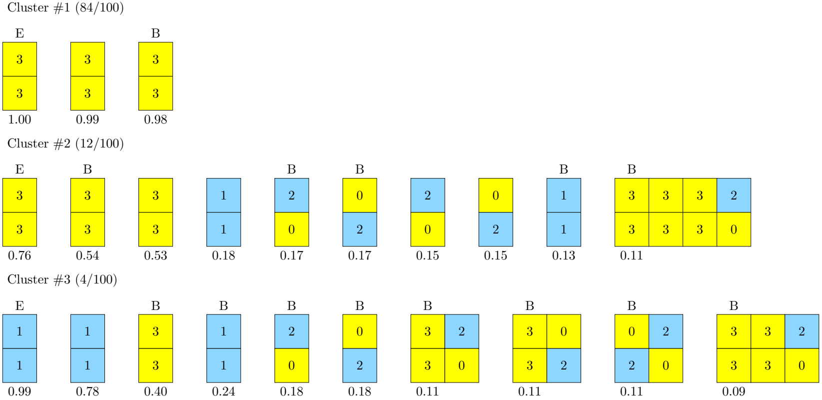


k-Means Patterns – Prisoner's Dilemma

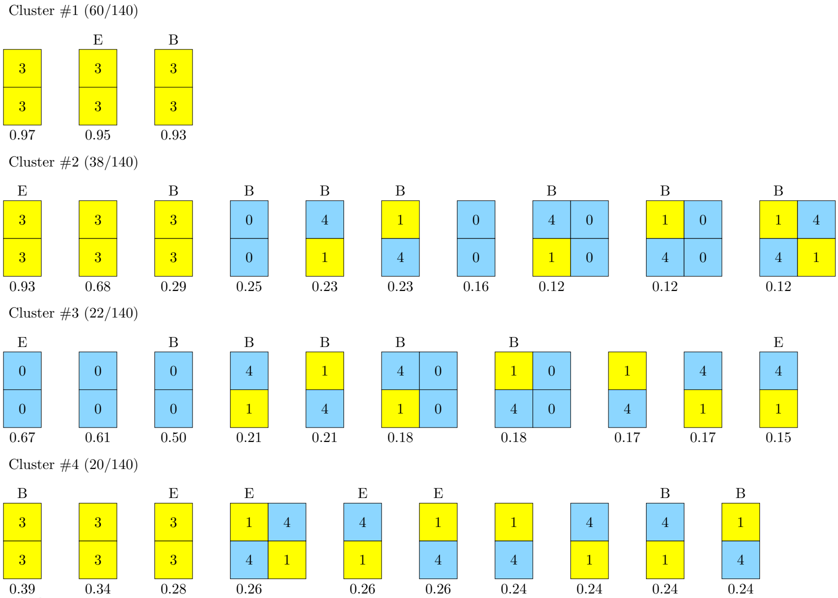


k-Means Patterns – Battle of the Sexes



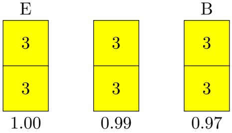

Cluster #1 (84/100)
E
B
3 3
3 3
3 3
1.00
0.99
0.98
Cluster #2 (12/100)
E
B
B
B
B
B
3 3
3 3
3 3
1 1
2 0
0 2
2 0
0 2
1 1
3 3 3 2
3 3 3 0
0.76
0.54
0.53
0.18
0.17
0.17
0.15
0.15
0.13
0.11
Cluster #3 (4/100)
E
B
B
B
B
B
B
B
B
1 1
1 1
3 3
1 1
2 0
0 2
3 2
3 0
3 0
3 2
0 2
2 0
3 3 2
3 3 0
0.99
0.78
0.40
0.24
0.18
0.18
0.11
0.11
0.11
0.09

k-Means Patterns – Chicken

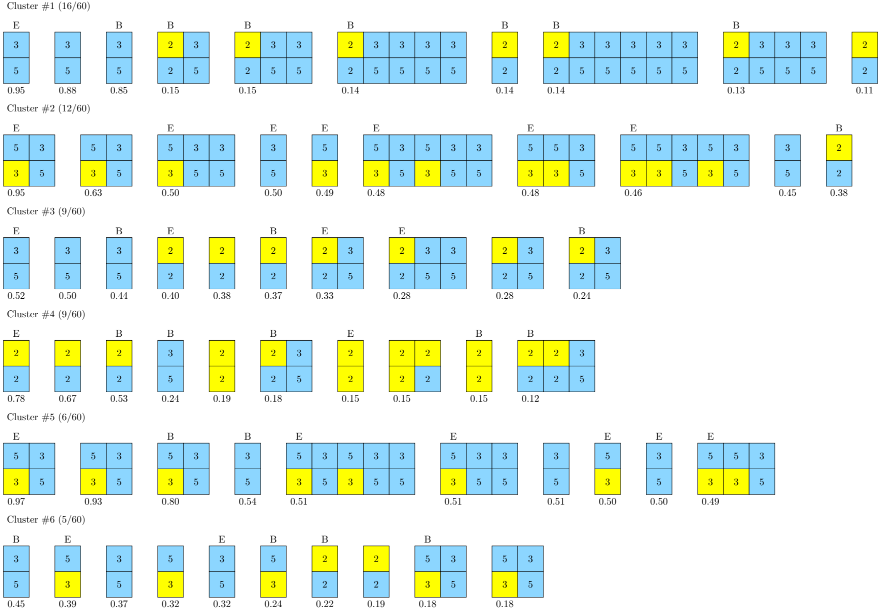


k-Means Patterns – Common Interest

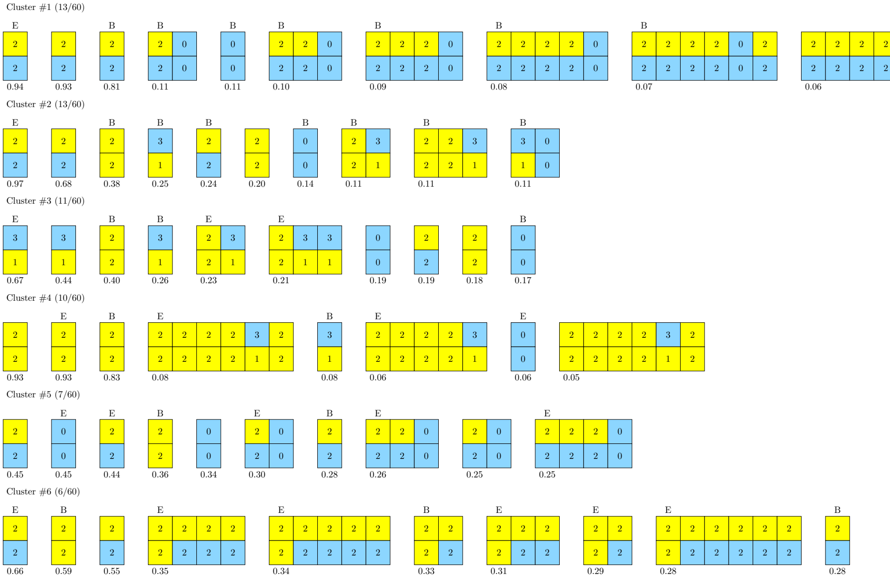
Cluster #1 (36/36)

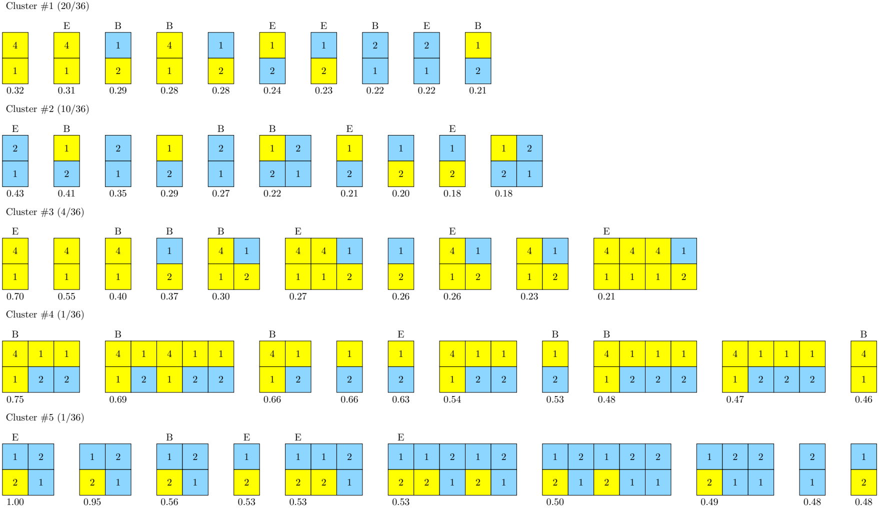


k-MEANS PATTERNS – SAMARITAN'S DILEMMA



k-Means Patterns – Ultimatum

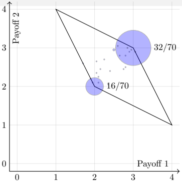



Cluster #1 (20/36)
0.32 0.31 0.29 0.28 0.28 0.24 0.23 0.22 0.22 0.21
Cluster #2 (10/36)
0.43 0.41 0.35 0.29 0.27 0.22 0.21 0.20 0.18 0.18
Cluster #3 (4/36)
0.70 0.55 0.40 0.37 0.30 0.27 0.26 0.26 0.23 0.21
Cluster #4 (1/36)
0.75 0.69 0.66 0.66 0.63 0.54 0.53 0.48 0.47 0.46
Cluster #5 (1/36)
1.00 0.95 0.56 0.53 0.53 0.53 0.50 0.49 0.48 0.48

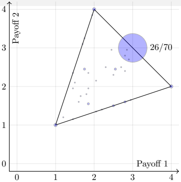
Concluding Remarks

- We uncover new regularities across games.
  - Lots of convergence to patterns.
  - Convergence can take time, especially with more complex patterns.
  - Egalitarianism and Pareto efficiency are strong attractors.
  - Learning predominantly leads to Pareto improvements and equality.
- These new regularities can be used to inform theory.
- Classify aspects of games to predict behavior in new games.

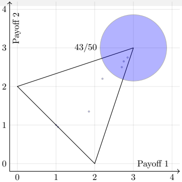
Heatmaps



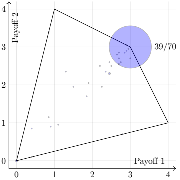
(a) Prisoner's Dilemma



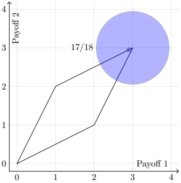
(b) Battle of the Sexes



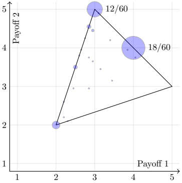
(c) Stag-Hunt



(d) Chicken



(e) Common Interest



(g) Samaritan's Dilemma