Repeated Games: The Prisoner's Dilemma

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Advertise	7-c,1	4-c,4-c

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Can cooperation emerge without external enforcement?

Repeated Game Example

- The unique Nash equilibrium is (D, D).

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Discounting

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  - This says, that I care more about my payoff in period 1 than in period 1,000.
- Let $\left(a^1, a^2, \ldots, a^T\right)$ be the choices for T periods; the player's discounted payoff is then,

$$u_i\left(a^1\right)+\delta u_i\left(a^2\right)+\delta^2 u_i\left(a^3\right)+\cdots+\delta^{T-1} u_i\left(a^T\right)=\sum_{t=1}^{T} \delta^{t-1} u_i\left(a^t\right).$$

Normalized Discounted Payoffs

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$$U_i\left(a^1, a^2, \ldots, a^T\right) = (1-\delta)\sum_{t=1}^{T}\delta^{t-1}u_i\left(a^t\right).$$

Repeated Game

Definition

Let G be a strategic game. Denote the set of players by N and the set of actions and payoff function of each player i by A_i and u_i, respectively. The T-period **repeated game** of G for discount factor δ is the extensive game with perfect information and simultaneous moves in which

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- the set of actions for player i after any history is A_i, and
- each player i evaluates terminal history according to normalized discounted payoff $U_i\left(a^1, a^2, \ldots, a^T\right)$.

Finitely Repeated Prisoner's Dilemma

- Consider the Prisoner's Dilemma game for $T = 2$.
  - How many terminal histories are there?
  - How many non-terminal histories are there?
  - How many strategies are there?
  - What are the the Nash equilibria?
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$$s_i\left(a^1,\ldots,a^t\right)=\begin{cases} C & h=\emptyset \\ C & \text{if } (a_j^1,\ldots,a_j^t)=(C,\ldots,C) \\ D & \text{otherwise.}\end{cases}$$

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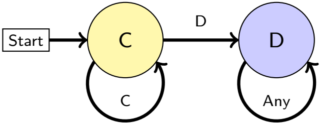

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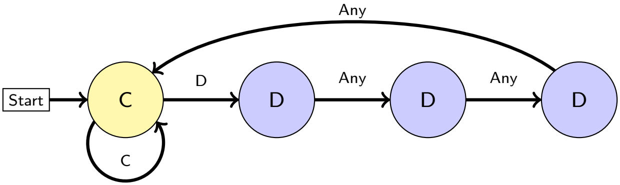

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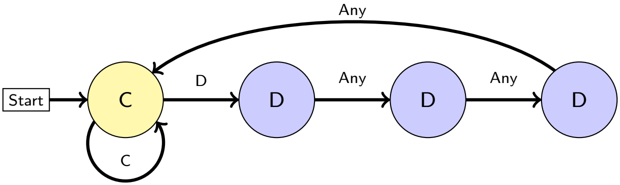

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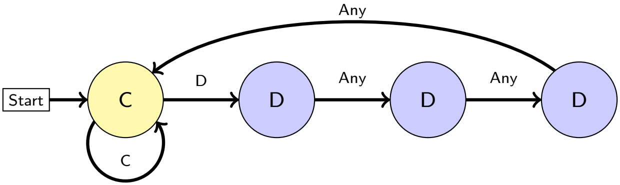

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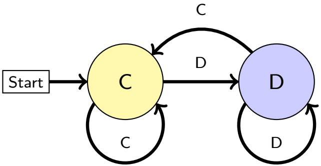

Examples of Tit-for-Tat

	High	Low
High	20 20	0 35
Low	35 0	10 10

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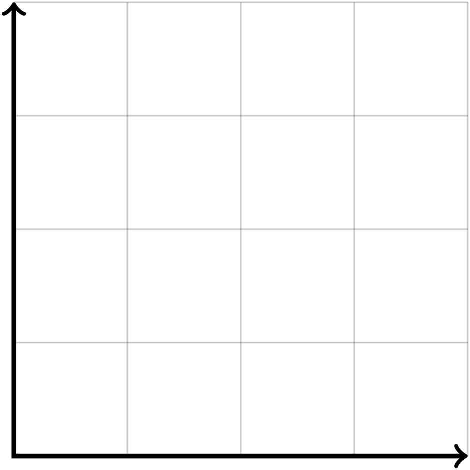
  1. I set a high price in the first period,
  2. if you lower your price, then, I will lower my price too,
  3. otherwise, if you keep your price high, I will keep my price high.

Feasible Payoffs



	C	D
C	2 2	0 3
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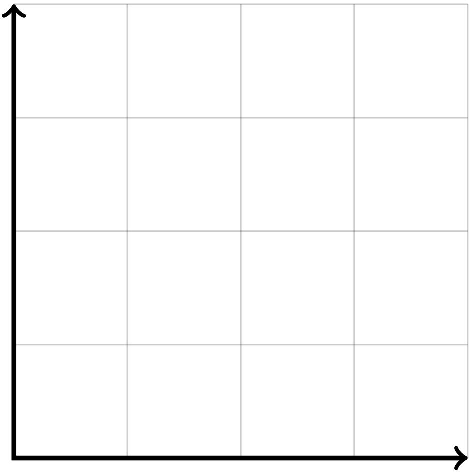

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What payoffs are possible as a Nash equilibrium?

Minmax Payoff

Definition

Player i's **minmax payoff** in a strategic game is

$$\underline{u}_i = \min_{a_{-i} \in A_{-i}} \left(\max_{a_i \in A_i} u_i \left(a_i, a_{-i}\right) \right).$$

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- If player 2 tries to punish player 1 forever, this is the best player 1 can do.

Example

- What would happen in this game?

	C	D
A	3,3	2,1
B	1,2	0,0

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Example

- What would happen in this game?
- In this game:
  - Action A is strictly dominant for Player 1.
  - Action C is strictly dominant for Player 2.

	C	D
A	3,3	2,1
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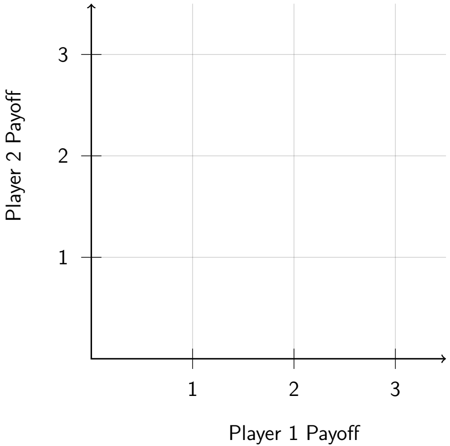
Example

- What would happen in this game?
- In this game:
  - Action A is strictly dominant for Player 1.
  - Action C is strictly dominant for Player 2.
- Players will repeatedly play (A, C).

	C	D
A	3,3	2,1
B	1,2	0,0

Example (Cont.)



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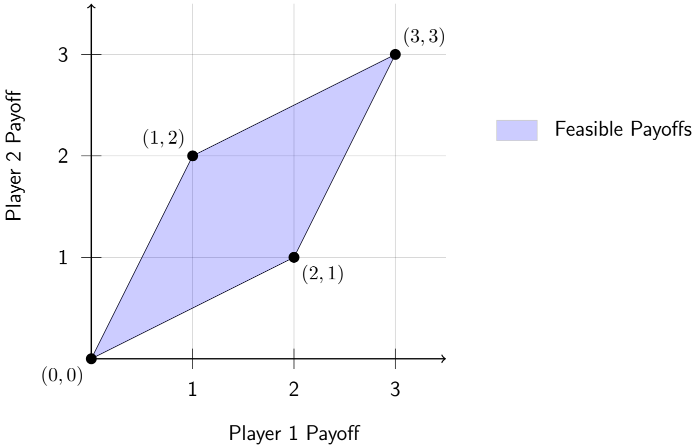

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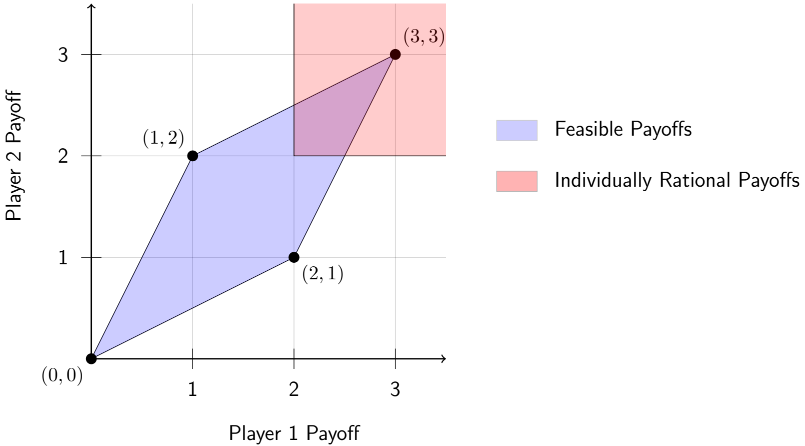

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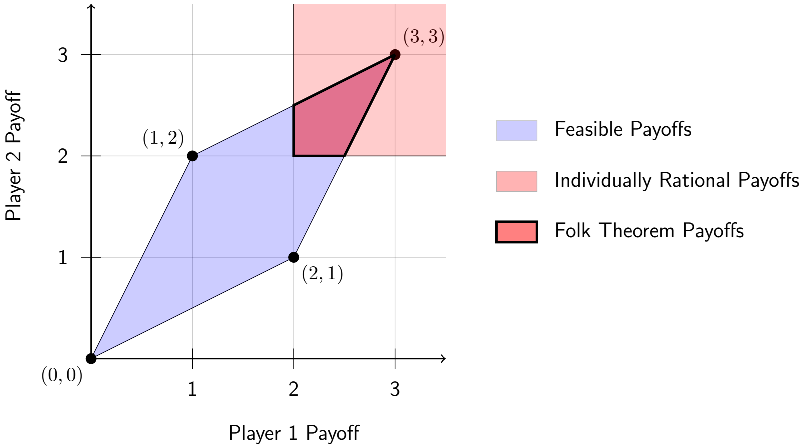

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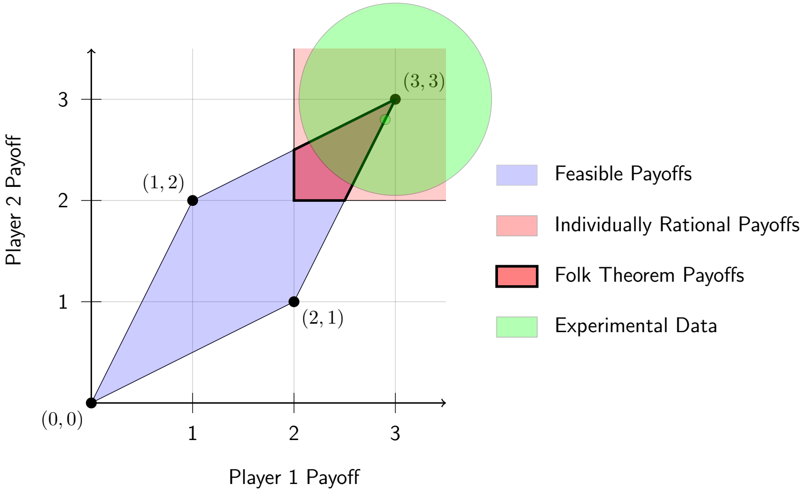

Minmax examples

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	C	D
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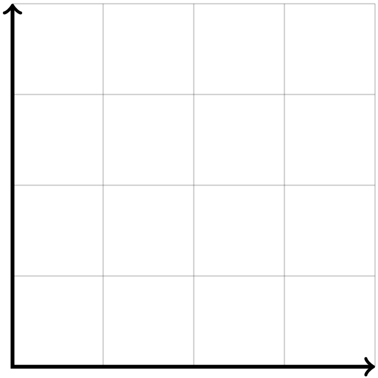
Folk Theorem



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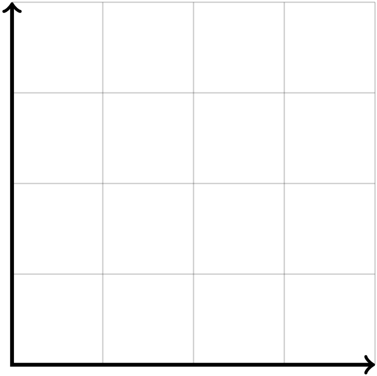

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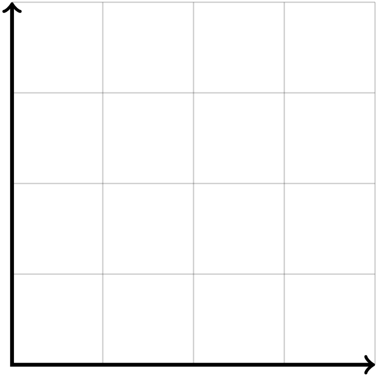

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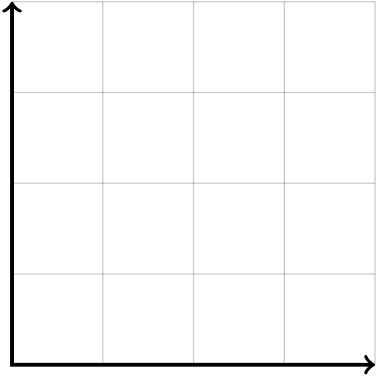
Folk Theorem

Any payoff (v_1, v_2) where both $v_1 > \underline{u}_1$ and $v_2 > \underline{u}_2$, can be supported as a Nash equilibrium if players are sufficiently patient.

Folk Theorem (Cont.)



	C	D
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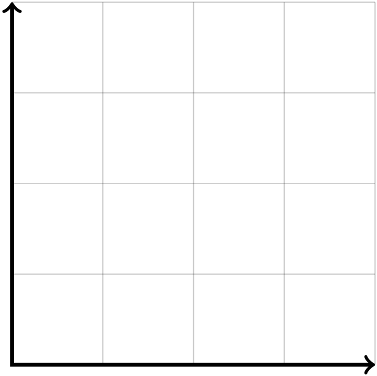

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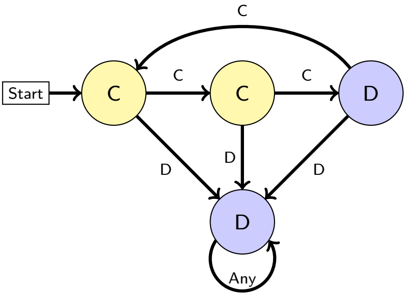

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